Quaternary Constant-Composition Codes with Weight Four and Distances Five or Six

Mingzhi Zhu and Gennian Ge

***Abstract*—The sizes of optimal constant-composition codes of weight three have been determined by Chee, Ge and Ling with four cases in doubt. Group divisible codes played an important role in their constructions. In this paper, we study the problem of constructing optimal quaternary constant-composition codes with Hamming weight four and minimum distances five or six through group divisible codes and Room square approaches. The problem is solved leaving only five lengths undetermined. Previously, the results on the sizes of such quaternary constant-composition codes were scarce.**

***Index Terms*—Constant-composition codes, group divisible codes, quaternary codes, Room square constructions.**

I. Introduction

Constant-composition codes (CCCs) are a special type of constant-weight codes (CWCs) which are important in coding theory. The class of constant-composition codes includes the important permutation codes and have attracted recent interest due to their numerous applications, such as in determining the zero error decision feedback capacity of discrete memoryless channels [38], multiple-access communications [14], spherical codes for modulation [23], DNA codes [9], [29], [32], powerline communications [10], [12], frequency hopping [11], frequency permutation arrays [28], and coding for bandwidth-limited channels [13].

Systematic study began in late 1990's [3], [5], [35]. Today, various methods have been applied to the problem of determining the maximum size of a constant-composition code, such as computer search methods [4], packing designs [11], [17], [18], [27], [39]–[42], tournament designs [43], polynomials and nonlinear functions [11], [15], [16], [19], [20], difference triangle sets [6], PBD-closure methods [7], [8] and some other methods [31], [36].

In the paper of Svanström et al. [37], some methods for providing upper and lower bounds on the maximum size $A_3(n, d, \overline{w})$ of a ternary code with length n, minimum Hamming distance d, and constant composition $\overline{w}$ were presented. The sizes of optimal ternary constant-composition codes with weight three have been determined completely by Chee, Ge and Ling in [7]. The sizes of optimal ternary constant-composition codes with weight four and distance five have been determined completely by Gao and Ge in [24].

The research of G. Ge was supported by the National Outstanding Youth Science Foundation of China under Grant No. 10825103, National Natural Science Foundation of China under Grant No. 61171198, Specialized Research Fund for the Doctoral Program of Higher Education, and Zhejiang Provincial Natural Science Foundation of China under Grant No. D7080064.

M. Zhu and G. Ge (Corresponding author. Email: gnge@zju.edu.cn) are with the Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, People's Republic of China.

The sizes of optimal quaternary constant-composition codes with weight three have been determined almost completely by Chee, Ge and Ling in [7] with four lengths in doubt. Recently, the problem of determining the sizes for optimal quaternary constant-composition codes with weight four and distance seven has been settled by Chee, Dau, Ling and Ling in [6]. In this paper, we will concentrate our attention on quaternary CCCs with weight four and distances five or six. The problem is solved leaving only five lengths undetermined for the case of distance five. Previously, the results on the sizes of such quaternary constant-composition codes were scarce.

II. Preliminaries

A. Definitions and Notations

The set of integers $\{i, i+1, \ldots, j\}$ is denoted by $[i, j]$. The ring $\mathbb{Z}/q\mathbb{Z}$ is denoted by $\mathbb{Z}_q$. The notation $\wr\cdot\wr$ is used for multisets.

All sets considered in this paper are finite if not obviously infinite. If X and R are finite sets, R^X denotes the set of vectors of length $|X|$, where each component of a vector $\mathsf{u} \in R^X$ has value in R and is indexed by an element of X, that is, $\mathsf{u} = (\mathsf{u}_x)_{x\in X}$, and $\mathsf{u}_x \in R$ for each $x \in X$. A *q-ary* code of *length* n is a set $\mathcal{C} \subseteq \mathbb{Z}_q^X$ for some X with size n. The elements of $\mathcal{C}$ are called *codewords*. The *Hamming norm* or the *Hamming weight* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$ is defined as $\|\mathsf{u}\| = |\{x \in X : \mathsf{u}_x \neq 0\}|$. The distance induced by this norm is called the *Hamming distance*, denoted d_H, so that $d_H(\mathsf{u}, \mathsf{v}) = \|\mathsf{u} - \mathsf{v}\|$, for $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_q^X$. The *composition* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$ is the tuple $\overline{w} = [w_1, \ldots, w_{q-1}]$, where $w_j = |\{x \in X : \mathsf{u}_x = j\}|$. For any two vectors $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_q^X$, define their *support* as $\mathrm{supp}(\mathsf{u}, \mathsf{v}) = \{x \in X : \mathsf{u}_x \neq \mathsf{v}_x\}$. We write $\mathrm{supp}(\mathsf{u})$ instead of $\mathrm{supp}(\mathsf{u}, \mathsf{0})$ and also call $\mathrm{supp}(\mathsf{u})$ the support of u.

A code $\mathcal{C}$ is said to have minimum distance d if $d_H(\mathsf{u}, \mathsf{v}) \geq d$ for all distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$. If $\|\mathsf{u}\| = w$ for every codeword $\mathsf{u} \in \mathcal{C}$, then $\mathcal{C}$ is said to be of (constant) *weight* w. A q-ary code $\mathcal{C}$ has *constant composition* $\overline{w}$ if every codeword in $\mathcal{C}$ has composition $\overline{w}$. A q-ary code of length n, distance d, and constant composition $\overline{w}$ is referred to as an $(n, d, \overline{w})_q$-code. The maximum size of an $(n, d, \overline{w})_q$-code is denoted as $A_q(n, d, \overline{w})$ and the $(n, d, \overline{w})_q$-codes achieving this size are called *optimal*. Note that the following operations do not affect distance and weight properties of an $(n, d, \overline{w})_q$-code:

(i) reordering the components of $\overline{w}$, and
(ii) deleting zero components of $\overline{w}$.

Consequently, throughout this paper, we restrict our attention to those compositions $\overline{w} = [w_1, \ldots, w_{q-1}]$, where $w_1 \geq \cdots \geq w_{q-1} \geq 1$.

Suppose $\mathsf{u} \in \mathbb{Z}_q^X$ is a codeword of an $(n, d, \overline{w})_q$-code, where $\overline{w} = [w_1, \ldots, w_{q-1}]$. Let $w = \sum_{i=1}^{q-1} w_i$. We can represent u equivalently as a w-tuple $\langle a_1, a_2, \ldots, a_w \rangle \in X^w$, where

$$\begin{aligned} \mathsf{u}_{a_1} = \cdots = \mathsf{u}_{a_{w_1}} &= 1, \\ \mathsf{u}_{a_{w_1+1}} = \cdots = \mathsf{u}_{a_{w_1+w_2}} &= 2, \\ &\vdots \\ \mathsf{u}_{a_{\sum_{i=1}^{q-2} w_i + 1}} = \cdots = \mathsf{u}_w &= q-1. \end{aligned}$$

Throughout this paper, we shall often represent codewords of constant-composition codes in this form. This has the advantage of being more succinct and more flexible in manipulation.

B. General Bounds

Lemma 2.1 (Chee et al. [7]):

$$A_q(n, d, [w_1, \ldots, w_{q-1}]) = \begin{cases} \binom{n}{\sum_{i=1}^{q-1} w_i} \binom{\sum_{i=1}^{q-1} w_i}{w_1, \ldots, w_{q-1}}, & \text{if } d \leq 2 \\ \left\lfloor \frac{n}{\sum_{i=1}^{q-1} w_i} \right\rfloor, & \text{if } d = 2\sum_{i=1}^{q-1} w_i \\ 1, & \text{if } d \geq 2\sum_{i=1}^{q-1} w_i + 1. \end{cases}$$

The following Johnson-type bound has been proven for constant-composition codes.

Lemma 2.2 (Svanström et al. [37]):

$$A_q(n, d, [w_1, \ldots, w_{q-1}]) \leq \frac{n}{w_1} A_q(n-1, d, [w_1-1, \ldots, w_{q-1}]).$$

Moreover, we have the following results.

Lemma 2.3 (Chee et al. [8]):

$$A_q(n, d, [w_1, \ldots, w_{q-1}]) \leq \begin{cases} \left\lfloor \frac{n}{w_1} \left\lfloor \frac{n-1}{\sum_{i=1}^{q-1} w_i - 1} \right\rfloor \right\rfloor & \text{if } d = 2\sum_{i=1}^{q-1} w_i - 3 \\ \left\lfloor \frac{n}{w_1} \left\lfloor \frac{n-1}{w_1 - 1} \right\rfloor \right\rfloor & \text{if } d = 2\sum_{i=1}^{q-1} w_i - 2. \end{cases}$$

Corollary 2.1:

$$A_4(n, 5, [2, 1, 1]) \leq n \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$A_4(n, 6, [2, 1, 1]) \leq \left\lfloor \frac{n}{2} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor.$$

Proof: The first equation follows by Lemma 2.2 and Lemma 2.1 where $w_1 = 1, w_2 = 2$ and $w_3 = 1$.

The second equation follows by Lemma 2.3 and Lemma 2.1 where $w_1 = 2, w_2 = 1$ and $w_3 = 1$. ■

In the following, we denote $U(n, 5, [2, 1, 1]) = n \left\lfloor \frac{n-1}{2} \right\rfloor$ and $U(n, 6, [2, 1, 1]) = \left\lfloor \frac{n}{2} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor$.

C. Designs

Our recursive construction is based on some combinatorial structures in design theory. The most important tools are pairwise balanced designs (PBDs) and group divisible designs (GDDs).

Let K be a subset of positive integers and let λ be a positive integer. A *pairwise balanced design* (PBD(v, K, λ) or (K, λ)-PBD) of order v with block sizes from K is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a finite set (the point set) of cardinality v and $\mathcal{B}$ is a family of subsets (blocks) of $\mathcal{V}$ that satisfy (1) if $B \in \mathcal{B}$, then $|B| \in K$ and (2) every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks of $\mathcal{B}$. The integer λ is the index of the PBD. The notations PBD(v, K) and K-PBD of order v are often used when $\lambda = 1$. If an element $k \in K$ is "starred" (written $k^\star$), it means that the PBD has exactly one block with size k.

Lemma 2.4 ([1]): For any integer $v \geq 10$, a $(v, \{4, 5, 6\}, 1)$-PBD exists with exceptions $v \in \{7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$.

Lemma 2.5 (Ling, Zhu, Colbourn, Mullin [30]): For any integer $v \geq 10$, a $(v, \{5, 6, 7, 8, 9\}, 1)$-PBD exists with exceptions $v \in [10, 20] \cup [22, 24] \cup [27, 29] \cup [32, 34]$.

Lemma 2.6 (Rees, Stinson [33]): There exists a $(v, \{4, w^\star\}, 1)$-PBD with $v > w$ if and only if $v \geq 3w + 1$, and:

(i) $v \equiv 1$ or $4 \pmod{12}$ and $w \equiv 1$ or $4 \pmod{12}$; or
(ii) $v \equiv 7$ or $10 \pmod{12}$ and $w \equiv 7$ or $10 \pmod{12}$.

Let K and G be sets of positive integers and let λ be a positive integer. A *group divisible design* of index λ and order v ((K, λ)-GDD) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (groups) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (blocks) of $\mathcal{V}$ that satisfy (1) if $B \in \mathcal{B}$ then $|B| \in K$, (2) every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks or one group, but not both, and (3) $|\mathcal{G}| > 1$. If $v = a_1 g_1 + a_2 g_2 + \cdots + a_s g_s$, and if there are a_i groups with size g_i, $i = 1, 2, \ldots, s$, then the (K, λ)-GDD is of type $g_1^{a_1} g_2^{a_2} \ldots g_s^{a_s}$. This is exponential notation for the group type. If $K = \{k\}$, then the (K, λ)-GDD is a (k, λ)-GDD. If $\lambda = 1$, the GDD is a K-GDD. Furthermore, a $(\{k\}, 1)$-GDD is a k-GDD. A *parallel class* or *resolution class* is a collection of blocks that partition the point-set of the design. A GDD is *resolvable* if the blocks of the design can be partitioned into parallel classes. A resolvable GDD is denoted by RGDD.

Lemma 2.7 ([25]): There exists a 4-RGDD of type g^u for each $(g, u) \in \{(3, 8), (4, 7)\}$.

Lemma 2.8 ([25]): There exists a 4-GDD of type $g^u m^1$ for each $(g, u, m) \in \{(3, 5, 0), (4, 6, 7), (12, 4, 18), (12, 5, 18), (15, 4, 21)\}$.

A $\{k\}$-GDD of type n^k is also called a *transversal design* and denoted by $\mathrm{TD}(k, n)$.

Lemma 2.9 ([2]): Let n be a positive integer. Then:

(i) a TD$(5,n)$ exists if $n \notin \{2,3,6,10\}$;
(ii) a TD$(6,n)$ exists if $n \notin \{2,3,4,6,10,14,18,22\}$;
(iii) a TD$(7,n)$ exists if $n \notin \{2,3,4,5,6,10,14,15,18,20,22,26,30,34,38,46,60\}$.
(iv) a TD$(8,9)$ exists.

D. Group Divisible Codes

Given $\mathsf{u} \in \mathbb{Z}_q^X$ and $Y \subseteq X$, the *restriction of* u *to* Y, written $\mathsf{u}\mid_Y$, is the vector $\mathsf{v} \in \mathbb{Z}_q^X$ such that

$$\mathsf{v}_x = \begin{cases} \mathsf{u}_x, & \text{if } x \in Y \\ 0, & \text{if } x \in X \backslash Y. \end{cases}$$

A *group divisible code* (GDC) of distance d is a triple $(X, \mathcal{G}, \mathcal{C})$, where $\mathcal{G} = \{G_1, \ldots, G_t\}$ is a partition of X with cardinality $|X| = n$ and $\mathcal{C} \subseteq \mathbb{Z}_q^X$ is a q-ary code of length n, such that $d_H(u,v) \geq d$ for each distinct $u, v \in \mathcal{C}$, and $\|u|_{G_i}\| \leq 1$ for each $u \in \mathcal{C}$, $1 \leq i \leq t$. Elements of $\mathcal{G}$ are called groups. We denote a GDC$(X, \mathcal{G}, \mathcal{C})$ of distance d as w-GDC(d) if $\mathcal{C}$ is of constant weight w. If we want to emphasize the composition of the codewords, we denote the GDC as $\overline{w}$-GDC(d) when every $u \in \mathcal{C}$ has composition $\overline{w}$. The type of a GDC$(X, \mathcal{G}, \mathcal{C})$ is the multiset $\wr |G| : G \in \mathcal{G} \wr$. As in the case of GDDs, the exponential notation is used to describe the type of a GDC. The size of a GDC$(X, \mathcal{G}, \mathcal{C})$ is $|\mathcal{C}|$. Note that an $(n, d, \overline{w})_q$-code with size s is equivalent to a $\overline{w}$-GDC(d) of type 1^n with size s.

Constant-composition codes of larger orders can often be obtained from GDCs via the following two constructions.

Construction 2.1 ((Filling in Groups) [7]): Let $d \leq 2(w-1)$. Suppose there exists a w-GDC(d) $(X, \mathcal{G}, \mathcal{C})$ of type $g_1^{t_1} \cdots g_s^{t_s}$ with size a. Suppose further that for each i, $1 \leq i \leq s$, there exists a $(g_i, d, w)_q$-code $\mathcal{C}_i$ with size b_i, then there exists a $(\sum_{i=1}^s t_i g_i, d, w)_q$-code $\mathcal{C}'$ with size $a + \sum_{i=1}^s t_i b_i$. In particular, if $\mathcal{C}$ and $\mathcal{C}_i$, $1 \leq i \leq s$, are of constant composition $\overline{w}$, then $\mathcal{C}'$ is also of constant composition $\overline{w}$.

Construction 2.2 ((Adjoining y Points) [7]): Let $y \in \mathbb{Z}_{\geq 0}$. Suppose there exists a (master) w-GDC(d) of type $g_1^{t_1} \cdots g_s^{t_s}$ with size a, and suppose the following (ingredients) also exist:

(i) a $(g_1 + y, d, w)_q$-code with size b,
(ii) a w-GDC(d) of type $1^{g_i} y^1$ with size c_i for each $2 \leq i \leq s$,
(iii) a w-GDC(d) of type $1^{g_1} y^1$ with size c_1 if $t_1 \geq 2$.

Then, there exists a $(y + \sum_{i=1}^s t_i g_i, d, w)_q$-code with size

$$a + b + (t_1 - 1)c_1 + \sum_{i=2}^{s} t_i c_i.$$

Furthermore, if the master and ingredient codes are of constant composition, then so is the resulting code.

The following two constructions are useful for generating GDCs of larger orders from smaller ones.

Construction 2.3 ((Fundamental Construction) [7]): Let $d \leq 2(w-1)$, $\mathcal{D} = (X, \mathcal{G}, \mathcal{A})$ be a (master) GDD, and $\omega : X \to \mathbb{Z}_{\geq 0}$ be a weight function. Suppose that for each $A \in \mathcal{A}$, there exists an (ingredient) w-GDC(d) of type $\wr \omega(a) : a \in A \wr$. Then there exists a w-GDC(d) $\mathcal{D}^*$ of type $\wr \sum_{x \in G} \omega(x) : G \in \mathcal{G} \wr$. Furthermore, if the ingredient GDCs are of constant composition $\overline{w}$, then $\mathcal{D}^*$ is also of constant composition $\overline{w}$.

Construction 2.4: Suppose there exists a w-GDC(d) of type $g_1^{t_1} \ldots g_s^{t_s}$ with size a. Suppose further that there exists a TD(w, m), then there exists a w-GDC(d) of type $(mg_1)^{t_1} \ldots (mg_s)^{t_s}$ with size am^2. If the original GDC is of constant composition $\overline{w}$, then so is the derived GDC.

III. Determining the Value of $A_4(n, 5, [2,1,1])$

A. Room Square Construction

Let S be a set of $n+1$ elements (symbols). A *Room square* of side n (on symbol set S), RS(n), is an $n \times n$ array, F, that satisfies the following properties:

1) every cell of F either is empty or contains an unordered pair of symbols from S,
2) each symbol of S occurs once in each row and column of F,
3) every unordered pair of symbols occurs in precisely one cell of F.

Lemma 3.1 ([34]): A Room square of side n exists if and only if n is odd and $n \neq 3$ or 5.

From each filled cell (r,c) of an RS(n), R, one can obtain an underlying 4-subset $\{i,j,r,c\}$, where $\{i,j\}$ occurs in column c and row r of R. A Room square of side n is called *super-simple* (denoted by SSRS(n)), if for any two filled cells (r_1, c_1) and (r_2, c_2) containing the symbols $\{i_1, j_1\}$ and $\{i_2, j_2\}$ respectively, the underlying 4-subsets $\{i_1, j_1, r_1, c_1\}$ and $\{i_2, j_2, r_2, c_2\}$ share at most two common elements.

Theorem 3.1: For an odd integer n, suppose there exists an SSRS(n), then there exists an optimal $(n, 5, [2,1,1])_4$-code with size $n(n-1)/2 = U(n, 5, [2,1,1])$.

Proof: For each filled cell (r,c) of the given SSRS(n), R, we form a codeword $\langle i, j, r, c \rangle$ of type $[2,1,1]$, where $\{i,j\}$ occurs in column c and row r of R. We have in total of $n(n-1)/2$ such codewords. Now, we prove that these $n(n-1)/2$ codewords form an optimal code of length n.

Because R is super-simple, so any two codewords intersect in at most two coordinates. If the distance between any two codewords, $\langle i_1, j_1, r_1, c_1 \rangle$ and $\langle i_2, j_2, r_2, c_2 \rangle$, is less than 5, then one of the following five properties must be satisfied:

1) $i_1 = i_2$ and $j_1 = j_2$,
2) $i_1 = j_2$ and $i_2 = j_1$,
3) $i_l = j_m$ and $r_1 = r_2$ where $l, m \in \{1,2\}$,
4) $i_l = j_m$ and $c_1 = c_2$ where $l, m \in \{1,2\}$, or
5) $r_1 = r_2$ and $c_1 = c_2$.

But any of these five properties has conflicts with the properties of a Room square. This means the distance between any two codewords is greater than or equal to 5. So these

codewords form an optimal code of length n with size $n(n-1)/2 = U(n,5,[2,1,1])$. ■

If $\{S_1,\ldots,S_n\}$ is a partition of a set S, an $\{S_1,\ldots,S_n\}$-*Room frame* is an $|S|\times|S|$ array, F, indexed by S, satisfying:

1) every cell of F either is empty or contains an unordered pari of symbols of S,
2) the subarrays $S_i \times S_i$ are empty, for each $1 \le i \le n$ (these subarrays are holes),
3) each symbol $x \notin S_i$ occurs once in row (or column) s for each $s \in S_i$, and
4) the pairs occurring in F are those $\{s,t\}$, where $(s,t) \in (S\times S)\setminus \cup_{i=1}^{n}(S_i\times S_i)$.

A Room square of side n is equivalent to a Room frame of type 1^n.

We can define a super-simple Room frame in the same way as a super-simple Room square.

Theorem 3.2: Suppose there exists a super-simple Room frame of type 2^t, then there exists an optimal $(2t,5,[2,1,1])_4$-code with size $2t(t-1) = U(2t,5,[2,1,1])$.

Proof: For each filled cell (r,c) of the super-simple Room frame of type 2^t, R, we form a codeword $\langle i,j,r,c\rangle$ of type $[2,1,1]$, where $\{i,j\}$ occurs in column c and row r of R. There are $2\times 2t(t-1)/2 = 2t(t-1)$ filled cells in R, which equals $U(2t,5,[2,1,1])$. Hence, we have in total of $U(2t,5,[2,1,1])$ such codewords. The rest of the proof is similar to that of Theorem 3.1. ■

A *starter* in the abelian group G of odd order (written additively), where $|G| = g$ is a set of unordered pairs $S = \{\{s_i,t_i\} : 1 \le i \le (g-1)/2\}$ that satisfies:

1) $\{s_i : 1 \le i \le (g-1)/2\} \cup \{t_i : 1 \le i \le (g-1)/2\} = G\setminus\{0\}$, and
2) $\{\pm(s_i - t_i) : 1 \le i \le (g-1)/2\} = G\setminus\{0\}$.

A *strong starter* is a starter $S = \{\{s_i,t_i\}\}$ in the abelian group G with the additional property that $s_i + t_i = s_j + t_j$ implies $i = j$, and for each i, $s_i + t_i \ne 0$. Let $S = \{\{s_i,t_i\} : 1 \le i \le (g-1)/2\}$ and $T = \{\{u_i,v_i\} : 1 \le i \le (g-1)/2\}$ be two starters in G. Without loss of generality, assume that $s_i - t_i = u_i - v_i$, for all i. Then S and T are called *orthogonal* starters if $u_i - s_i = u_j - s_j$ implies $i = j$, and if $u_i \ne s_i$ for all i. An *adder* for the starter $S = \{\{s_i,t_i\} : 1 \le i \le (g-1)/2\}$ is an ordered set $A_S = \{a_1,a_2,\ldots,a_{(g-1)/2}\}$ of $(g-1)/2$ distinct nonzero elements from G such that the set $T = \{\{s_i + a_i, t_i + a_i\} : 1 \le i \le (g-1)/2\}$ is also a starter in the group G. These two starters S and T are orthogonal starters.

Theorem 3.3 ([21]): If there exist two orthogonal starters in a group of order n, then there exists a Room square of side n. If the group is $\mathbb{Z}_n$, then the resulting Room square is cyclic.

Let G be an additive abelian group of order g, and let H be a subgroup of order h of G, where $g-h$ even. An *h-frame starter* of order g/h in $G\setminus H$ is a set of unordered pairs $S = \{\{s_i,t_i\}, 1 \le i \le (g-h)/2\}$ such that

1) $\{s_i : 1 \le i \le (g-h)/2\} \cup \{t_i : 1 \le i \le (g-h)/2\} = G\setminus H$, and
2) $\{\pm(s_i - t_i) : 1 \le i \le (g-h)/2\} = G\setminus H$.

A frame starter $A = \{\{s_i,t_i\}\}$ is strong if $s_i + t_i = s_j + t_j$ implies $i = j$, and $s_i + t_i \notin H$ for all i. Let $A = \{\{s_i,t_i\}\}$ and $B = \{\{u_i,v_i\}\}$ be two frame starters. We may assume that $t_i - s_i = v_i - u_i$, for each $1 \le i \le (g-t)/2$. We say that A and B are orthogonal frame starters if $u_i - s_i = u_j - s_j$ implies $i = j$, and $u_i - s_i \notin H$ for all i.

Lemma 3.2 ([22]): If $A = \{\{s_i,t_i\}\}$ is a strong frame starter then A and $-A = \{\{-s_i,-t_i\}\}$ are orthogonal frame starters.

Lemma 3.3 ([22]): If there exists a pair of orthogonal t-frame starters in $G\setminus H$ with $|G| = g$ and $|H| = t$, then there exists a Room frame of type t^u, where $u = g/t$.

So if we have a strong starter in a group of order n which can generate an SSRS(n), then we get an optimal $(n,5,[2,1,1])_4$-code with size $U(n,5,[2,1,1])$. Similarly, if we have a strong frame starter in $G\setminus H$ with $|G| = g$ and $|H| = 2$ which can generate a super-simple Room frame of type $2^{g/2}$, then we get an optimal $(g,5,[2,1,1])_4$-code with size $U(g,5,[2,1,1])$.

B. Some Small $[2,1,1]$-GDC(5)s and Optimal Codes with Distance 5

In the sequel, we construct some small $[2,1,1]$-GDC(5)s and optimal codes via computer search. The constructions are based on the familiar difference method, where a finite group (mostly abelian group $\mathbb{Z}_u$) will be utilized to generate all the codewords of a code or a GDC. Thus, instead of listing all the codewords, we list a set of base codewords and generate the others by an additive group and perhaps some further automorphisms. Mostly, the set of base codewords are divided into two parts, P and R, where each codeword of P will be multiplied by m^i for each $0 \le i \le s-1$ to generate s codewords, and R is the set of the remaining base codewords. The desired codes are generated by developing the base codewords $+M$ modulo n. Then, we just need to list n, m, s, M, P and R for each code. Sometimes, R may be empty, which will be omitted.

Proposition 3.1: There exists a $[2,1,1]$-GDC(5) of type g^t with size $\frac{g^2t(t-1)}{2}$ for the following parameters:

1) $g = 2$, $t \in \{8,9,11\}$,
2) $g = 3$, $t \in \{7,9\}$,
3) $g = 4$, $t \in \{5,6,7,8,9,11\}$,
4) $g = 6$, $t \in \{5\}$.

Proof: For each given pair $\{g,t\}$, let $X_{\{g,t\}} = \mathbb{Z}_{gt}$, $\mathcal{G}_{\{g,t\}} = \{\{i, t+i, \ldots, (g-1)t+i\} : i \in \mathbb{Z}_t\}$ and $\mathcal{C}_{\{g,t\}}$ be the set of cyclic (or quasi-cyclic) shifts of the vectors generated by the following vectors respectively. Then $(X_{\{g,t\}}, \mathcal{G}_{\{g,t\}}, \mathcal{C}_{\{g,t\}})$ is a $[2,1,1]$-GDC(5) of type g^t with size $\frac{g^2t(t-1)}{2}$. In order to save space, we list only one example

here. Other cases can be found in Propositions 6.1–6.4. For $g = 2$ and $t = 8$, we have $n = 16$, $m = 11$, $s = 2$, $M = 2$ and

$$\begin{array}{lllll} P: & \langle 0,13,15,6\rangle & & & \\ R: & \langle 1,11,0,13\rangle & \langle 0,1,4,14\rangle & \langle 0,11,9,15\rangle & \langle 0,3,7,10\rangle \\ & \langle 0,6,2,1\rangle & \langle 0,2,13,7\rangle & \langle 1,5,2,11\rangle & \langle 0,7,14,12\rangle \\ & \langle 0,5,1,3\rangle & \langle 0,9,3,4\rangle & \langle 0,12,6,9\rangle & \langle 1,3,12,2\rangle. \end{array}$$

■

Proposition 3.2: There exists a $[2,1,1]$-GDC(5) of type 4^t2^1 with size $8t^2$ for each $t \in \{5,6\}$.

Proof: Detailed constructions can be found in Propositions 6.5. ■

Proposition 3.3: $A_4(n,5,[2,1,1]) = U(n,5,[2,1,1])$ for each $n \in \{12,14,15,17,20,28,30,36,38,44,46,52,54,62,68,70,76,78,86,92,94,110,126,134\}$.

Proof: Detailed constructions can be found in Propositions 6.6. ■

Proposition 3.4: $A_4(n,5,[2,1,1]) = U(n,5,[2,1,1])$ for each $n \in \{19,21,23,24,25,26,27,29,31,32,33,34,35,37,39,40,41,42,43,45,47,48,49,50,51,53,55,56,57,58,59,61,63,64,65,66,67,69,71,73,74,75,77,79,83,85,87,88,89,93,95,99,103,104,106,107,109,111,123,125,127,131,133,138,139\}$.

Proof: All these optimal codes are constructed by strong starters or strong frame starters with n odd or even respectively. By Theorems 3.1 and 3.2, there exists an optimal code of such length n. The starters are given in a similar way as the codewords in the above propositions. In order to save space, we list only one example here. Other cases can be found in Proposition 6.7. For $n = 19$, we have $m = 4$, $s = 9$, $M = 1$ and $P : \{1,3\}$. ■

C. *The Case of Length* $n \equiv 0 \pmod 4$

Lemma 3.4: $A_4(4,5,[2,1,1]) = 1$ and $A_4(8,5,[2,1,1]) \geq 18$.

Proof: For $n = 4$, the one required codeword is $\langle 0,1,2,3\rangle$. For $n = 8$, the 18 required codewords can be found in Proposition 6.8. ■

Theorem 3.4: There exists an optimal $(4t,5,[2,1,1])_4$-code with size $4t(2t-1)$ for all $t \geq 3$.

Proof: For each $3 \leq t \leq 14$ or $t \in \{16,17,19,22,23,26\}$, there exists an optimal $(4t,5,[2,1,1])_4$-code by Propositions 3.1, 3.3 and 3.4.

For each $u \in \{5,6,7,9,11\}$, there exists a $[2,1,1]$-GDC(5) of type 4^u. Apply Construction 2.4 with a TD$(4,3)$ (which exists by Lemma 2.9) to get a $[2,1,1]$-GDC(5) of type 12^u. Filling in the groups of this GDC with an optimal $(12,5,[2,1,1])_4$-code coming from Proposition 3.3, the result is an optimal $(4t,5,[2,1,1])_4$-code for each $t \in \{15,18,21,27,33\}$.

For each $u \in \{7,8\}$, there exists a $[2,1,1]$-GDC(5) of type 4^u, and apply Construction 2.4 with a TD$(4,4)$ to get a $[2,1,1]$-GDC(5) of type 16^u for each $u \in \{7,8\}$. Filling in the groups of this GDC with an optimal $(16,5,[2,1,1])_4$-code (see Proposition 3.1), the result is an optimal $(4t,5,[2,1,1])_4$-code for each $t \in \{28,32\}$.

For $t = 31$, take a 4-RGDD of type 3^8 (see Lemma 2.7). There are 7 parallel classes in this 4-RGDD. Add one ideal point to each of these 7 parallel classes to complete them. The result is a 5-GDD of type 3^87^1. Apply Construction 2.3 with weight 4 to each point of this 5-GDD and fill in the groups with optimal codes of lengths 12 and 28 to obtain an optimal $(124,5,[2,1,1])$-code. Here, the input design, a $[2,1,1]$-GDC(5) of type 4^5, exists by Proposition 3.1.

For all $t \geq 34$ or $t \in \{20,24,25,29,30\}$, take a $(t+1,\{5,6,7,8,9\},1)$-PBD from Lemma 2.5 and remove one point to obtain a $\{5,6,7,8,9\}$-GDD of type $4^i5^j6^k7^l8^m$ with $4i+5j+6k+7l+8m = t$. Apply Construction 2.3 with weight 4 using $[2,1,1]$-GDC(5)s of types 4^t for $t \in \{5,6,7,8,9\}$ (Proposition 3.1) as input ingredients to obtain a $[2,1,1]$-GDC(5) of type $16^i20^j24^k28^l32^m$ and length $4t$. Filling in the groups of this GDC with optimal $(n,5,[2,1,1])_4$-codes for $n \in \{16,20,24,28,32\}$ by Propositions 3.1, 3.3 and 3.4, the result is an optimal $(4t,5,[2,1,1])_4$-code for all $t \geq 34$ or $t \in \{20,24,25,29,30\}$. ■

D. *The Case of Length* $n \equiv 1 \pmod 4$

Lemma 3.5: $A_4(5,5,[2,1,1]) = 2$, $A_4(9,5,[2,1,1]) \geq 27$ and $A_4(13,5,[2,1,1]) \geq 72$.

Proof: All the required codewords can be found in Proposition 6.9. ■

Theorem 3.5: There exists an optimal $(4t+1,5,[2,1,1])_4$-code with size $2t(4t+1)$ for all $t \geq 4$.

Proof: For each $4 \leq t \leq 19$ or $t \in \{21,22,23,27,31,33\}$, there exists an optimal $(4t+1,5,[2,1,1])_4$-code by Propositions 3.3 and 3.4.

For $t = 26$, there exists a $[2,1,1]$-GDC(5) of type 3^7. Apply Construction 2.4 with a TD$(4,5)$ to get a $[2,1,1]$-GDC(5) of type 15^7. Fill in the groups with an optimal $(15,5,[2,1,1])_4$-code to obtain an optimal $(4t+1,5,[2,1,1])_4$-code.

For each $u \in \{7,8\}$, there exists a $[2,1,1]$-GDC(5) of type 4^u. Apply Construction 2.4 with a TD$(4,4)$ to get a $[2,1,1]$-GDC(5) of type 16^u. Adjoin one ideal point to this GDC, and fill in the groups together with the extra point with an optimal $(17,5,[2,1,1])_4$-code to obtain an optimal $(4t+1,5,[2,1,1])_4$-code for each $t \in \{28,32\}$.

For all $t \geq 34$ or $t \in \{20,24,25,29,30\}$, take a $(t+1,\{5,6,7,8,9\},1)$-PBD from Lemma 2.5 and remove one point to obtain a $\{5,6,7,8,9\}$-GDD of type $4^i5^j6^k7^l8^m$ with $4i+5j+6k+7l+8m = t$. Apply construction 2.3 with weight 4 using $[2,1,1]$-GDC(5)s of types 4^t for $t \in \{5,6,7,8,9\}$ (Lemma 3.1) as input ingredients to obtain a $[2,1,1]$-GDC(5) of type $16^i20^j24^k28^l32^m$ and length $4t$. Adjoin one ideal point to this GDC, and fill in the groups together with the extra point with optimal $(n,5,[2,1,1])_4$-codes for $n \in \{17,21,25,29,33\}$ (Propositions 3.3 and 3.4) to obtain an optimal $(4t+1,5,[2,1,1])_4$-code for all $t \geq 34$ or $t \in \{20,24,25,29,30\}$. ■

E. The Case of Length $n \equiv 2 \pmod 4$

Lemma 3.6: $A_4(6,5,[2,1,1]) = 6$, $A_4(10,5,[2,1,1]) \geq 36$.

Proof: All the required codewords can be found in Proposition 6.10. ■

Theorem 3.6: There exists an optimal $(4t+2,5,[2,1,1])_4$-code with size $2t(4t+2)$ for each $3 \leq t \leq 37$.

Proof: For each $3 \leq t \leq 34$ and $t \notin \{20,22,24,25,28,29,30,32\}$, there exists an optimal $(4t+2,5,[2,1,1])_4$-code with size $2t(4t+2)$ by Propositions 3.1, 3.3 and 3.4.

For each $t \in \{20,24,28,32,36\}$, there exists a $[2,1,1]$-GDC(5) of type 4^u with $u \in \{5,6,7,8,9\}$. Apply Construction 2.4 with a TD$(4,4)$ to get a $[2,1,1]$-GDC(5) of type 16^u with $u \in \{5,6,7,8,9\}$. Adjoin two additional points and fill in the groups together with the two extra points with a $[2,1,1]$-GDC(5) of type 2^9 to obtain an optimal $[2,1,1]$-GDC(5) of type 2^{8u+1} and lengths $82, 98, 114, 130$ or 146.

For each $t \in \{22,37\}$, there exists a $[2,1,1]$-GDC(5) of type 6^5. Let $t_1 = (4t+2)/30$. Apply Construction 2.4 with a TD$(4,t_1)$ to get a $[2,1,1]$-GDC(5) of type $(6t_1)^5$. Fill in the groups with an optimal $(6t_1,5,[2,1,1])_4$-code to obtain an optimal code of length $4t+2$.

For each $t \in \{25,30,35\}$, there exists a $[2,1,1]$-GDC(5) of type 4^u with $u \in \{5,6,7\}$. Apply Construction 2.4 with a TD$(4,5)$ to get a $[2,1,1]$-GDC(5) of type 20^u with $u \in \{5,6,7\}$. Adjoin two additional points and fill in the groups together with the two extra points with a $[2,1,1]$-GDC(5) of type 2^{11} to obtain an optimal $[2,1,1]$-GDC(5) of type 2^{10u+1} and lengths $102, 122$ or 142.

For $t = 29$, take a TD$(6,5)$ from Lemma 2.9. Apply Construction 2.3 with weight 4 to the points in the first five groups and 4 points in the last group, and weight 2 to the other 1 points in the last group. All the remaining points are given weight 0. Note that there exist $[2,1,1]$-GDC(5)s of types 4^5, 4^6 and $4^5 2^1$ by Propositions 3.1 and 3.2. The result is a $[2,1,1]$-GDC(5) of type $20^5 18^1$. Filling in the groups with optimal codes of lengths 18 or 20, the results are optimal codes of length $118 = 4 \times 29 + 2$. ■

Theorem 3.7: There exists an optimal $(4t+2,5,[2,1,1])_4$-code with size $2t(4t+2)$ for all $t \geq 38$.

Proof: Take a TD$(7,r)$ for $r \geq 7$ and $r \notin \{10,14,15,18,20,22,26,30,34,38,46,60\}$ from Lemma 2.9. Apply Construction 2.3 with weight 4 to the points in the first five groups, x points in the sixth group and y points in the last group, and weight 2 to the other z points in the last group. All the remaining points are given weight 0. Here, we require that $x \geq 4$, $4y + 2z \geq 14$ and z odd. Note that there exist $[2,1,1]$-GDC(5)s of types 4^5, 4^6, 4^7, $4^5 2^1$, and $4^6 2^1$ by Propositions 3.1 and 3.2. The result is a $[2,1,1]$-GDC(5) of type $(4r)^5(4x)^1(4y+2z)^1$. Fill in the groups with optimal codes of lengths $4u$ with $u \geq 4$ (which exist by Theorem 3.4) or $4v+2$ with $3 \leq v \leq 37$ (which exist by Theorem 3.6). The result is an optimal $(20r+4x+4y+2z,5,[2,1,1])_4$-code, where $20r+4x+4y+2z$ can take any value n with $n \equiv 2 \pmod 4$ and $n \geq 4 \times 38 + 2 = 154$. ■

F. The Case of Length $n \equiv 3 \pmod 4$

Lemma 3.7: $A_4(7,5,[2,1,1]) = 10$, $A_4(11,5,[2,1,1]) \geq 48$.

Proof: All the required codewords can be found in Proposition 6.11. ■

Theorem 3.8: There exists an optimal $(4t+3,5,[2,1,1])_4$-code with size $(2t+1)(4t+3)$ for each $3 \leq t \leq 37$.

Proof: For each $3 \leq t \leq 34$ and $t \notin \{22,28,29,33\}$, there exists an optimal $(4t+3,5,[2,1,1])_4$-code with size $(2t+1)(4t+3)$ by Propositions 3.3 and 3.4.

For $t = 22$, there exists a $[2,1,1]$-GDC(5) of type 6^5. Apply Construction 2.4 with a TD$(4,3)$ to get a $[2,1,1]$-GDC(5) of type 18^5. Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(19,5,[2,1,1])_4$-code to obtain an optimal code of length $91 = 4 \times 22 + 3$.

For each $t \in \{28,29\}$, take a TD$(6,5)$ from Lemma 2.9. Apply Construction 2.3 with weight 4 to the points in the first five groups and x points in the last group, and weight 2 to the other y points in the last group. All the remaining points are given weight 0. Note that there exist $[2,1,1]$-GDC(5)s of types 4^5, 4^6 and $4^5 2^1$ by Propositions 3.1 and 3.2. The result is a $[2,1,1]$-GDC(5) of type $20^5(4x+2y)^1$. Here, $4x+2y$ can take 14 or 18 when $x = 3, y = 1$ or $x = 4, y = 1$. Adjoining one ideal point and filling in the groups together with the extra point with optimal codes of lengths 15, 19 or 21, the results are optimal codes of lengths $115 = 4 \times 28 + 3$ or $119 = 4 \times 29 + 3$.

For $t = 33$, there exists a $[2,1,1]$-GDC(5) of type 3^9 by Proposition 3.1. Apply Construction 2.4 with a TD$(4,5)$ to get a $[2,1,1]$-GDC(5) of type 15^9. Fill in the groups with an optimal $(15,5,[2,1,1])_4$-code to obtain an optimal code of length $135 = 4 \times 33 + 3$.

For $t = 35$, take a TD$(9,8)$ from Lemma 2.9. Apply Construction 2.3 with weight 2 to the points in the first eight groups and 7 points in the last group. The other points are given weight 0. Note that there exist $[2,1,1]$-GDC(5)s of types 2^8, 2^9 by Proposition 3.1. The result is a $[2,1,1]$-GDC(5) of type $16^8 14^1$. Adjoining one ideal point and filling in the group together with the extra point with optimal codes of lengths 15 or 17, the result is an optimal code of length $143 = 4 \times 35 + 3$.

For $t = 36$, there exists a $[2,1,1]$-GDC(5) of type 3^7 by Proposition 3.1. Apply Construction 2.4 with a TD$(4,7)$ to get a $[2,1,1]$-GDC(5) of type 21^7. Fill in the groups with an optimal $(21,5,[2,1,1])_4$-code to obtain an optimal code of length $147 = 4 \times 36 + 3$.

For $t = 37$, there exists a $[2,1,1]$-GDC(5) of type 6^5. Apply Construction 2.4 with a TD$(4,5)$ to get a $[2,1,1]$-GDC(5) of type 30^5. Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(31,6,[2,1,1])_4$-code to obtain an optimal code of length $151 = 4 \times 37 + 3$. ■

Theorem 3.9: There exists an optimal $(4t+3,5,[2,1,1])_4$-code with size $2t(4t+3)$ for all $t \geq 38$.

Proof: Take a TD$(7,r)$ for $r \geq 7$ and $r \notin \{10,14,15,18,20,22,26,30,34,38,46,60\}$ from Lemma 2.9. Apply Construction 2.3 with weight 4 to the points in the first five groups,

x points in the sixth group and y points in the last group, and weight 2 to the other z points in the last group. All the remaining points are given weight 0. We require that $x \geq 4$ and $4y+2z \geq 14$. Note that there exist $[2,1,1]$-GDC(5)s of types 4^5, 4^6, 4^7, $4^5 2^1$, and $4^6 2^1$ by Propositions 3.1 and 3.2. The result is a $[2,1,1]$-GDC(5) of type $(4r)^5(4x)^1(4y+2z)^1$. Adjoin one ideal point and fill in the groups together with the extra point with optimal codes of lengths $4u+1$ with $u \geq 4$ (which exist by Theorem 3.4) or $4v+3$ with $3 \leq v \leq 37$ (which exist by Theorem 3.8). The result is an optimal $(20r+4x+4y+2z+1, 5, [2,1,1])_4$-code, where $20r+4x+4y+2z+1$ can take any value of $4t+3$ with t greater than 38. ■

IV. Determining the Value of $A_4(n, 6, [2,1,1])$

A. Some Small $[2,1,1]$-GDC(6) and Optimal Codes with Distance 6

First, we construct some small $[2,1,1]$-GDC(6)s and optimal codes via computer search. In the codes with infinite points, the subscripts on the elements $x_0 \in \{x\} \times \mathbb{Z}_u$ for $x \in \{a,b,c,d,e\}$ are developed modulo the unique subgroup in the abelian group $\mathbb{Z}_n$ of order u.

Proposition 4.1: There exists a $[2,1,1]$-GDC(6) of type g^t with size $\frac{g^2 t(t-1)}{6}$ for the following parameters:

1) $g=2$, $t \in \{10,13,16,19,22,25,28,34\}$,
2) $g=3$, $t \in \{5,7\}$,
3) $g=4$, $t \in \{4,7\}$,
4) $g=6$, $t \in \{4,5,6,7\}$,
5) $g \in \{7,10,13,22\}$, $t \in \{4\}$.

Proof: Detailed constructions can be found in Propositions 7.1–7.6, 7.9 and 7.10. ■

Proposition 4.2: There exists a $[2,1,1]$-GDC(6) of type $12^t u^1$ with size $12t(2t+\frac{u}{3}-2)$ for the following parameters:

1) $u=9$, $t \in \{4,5,\ldots,15,17,18,19,23\}$,
2) $u=15$, $t \in \{7,8,\ldots,15\}$.

Proof: Detailed constructions can be found in Propositions 7.7 and 7.8. ■

Proposition 4.3: There exists a $[2,1,1]$-GDC(6) of type $1^{12}2^1$ with size 28.

Proof: Detailed construction can be found in Proposition 7.11. ■

Proposition 4.4: There exists a $[2,1,1]$-GDC(6) of type $1^t 11^1$ with size $6u^2+20u$, where $6u=t$ for each $t \in \{30,36,54,66,78\}$.

Proof: Detailed constructions can be found in Proposition 7.12. ■

Proposition 4.5: $A_4(n,6,[2,1,1]) = U(n,6,[2,1,1])$ for each $n \in \{6,8,10,11,13,14,16,17,19,22,23,25,28,31,34,35,37,43,55,67,79,103\}$.

Proof: Detailed constructions can be found in Proposition 7.13. ■

B. The Case of Length $n \equiv 0, 1 \pmod 6$

Lemma 4.1: $A_4(7,6,[2,1,1]) = 4$.

Proof: All the required codewords can be found in Proposition 7.14. ■

Lemma 4.2: For any positive integer t, if $A_4(6t+1,6,[2,1,1]) = U(6t+1,6,[2,1,1])$ then $A_4(6t,6,[2,1,1]) = U(6t,6,[2,1,1])$.

Proof: In an optimal $(6t+1,6,[2,1,1])_4$-code, every coordinate has exactly $2t+2t=4t$ non-zero elements. Fix a coordinate x and remove all the $4t$ codewords containing non-zero elements in this coordinate x. Shorten all the remaining codewords by deleting the element 0 in coordinate x from them. The resultant codewords form an optimal $(6t,6,[2,1,1])_4$-code with size $U(6t+1,6,[2,1,1]) - 4t = 6t^2-3t = U(6t,6,[2,1,1])$. ■

Lemma 4.3: There exists a $[2,1,1]$-GDC(6) of type 12^t for all $t \geq 4$.

Proof: When $t \equiv 0$ or $1 \pmod 4$ and $t \geq 4$, there exists a $(3t+1,\{4\},1)$-PBD by Lemma 2.6. Deleting one point from the point set gives a $\{4\}$-GDD of type 3^t. When $t \equiv 2$ or $3 \pmod 4$ and $t \geq 7$, there exists a $(3t+1,\{4,7^\star\},1)$-PBD by Lemma 2.6. Remove one point from this PBD which is not in the unique block with size 7 to obtain a $\{4,7^\star\}$-GDD of type 3^t. Hence, we always have a $\{4,7\}$-GDD of type 3^t for all $t \geq 4$ and $t \neq 6$.

Apply Construction 2.3 with weight 4 to obtain a $[2,1,1]$-GDC(6) of type 12^t for all $t \geq 4$ and $t \neq 6$. Here, the input $[2,1,1]$-GDC(6)s of types 4^4 and 4^7 exist by Proposition 4.1.

For $t=6$, take a $\{5\}$-GDD of type 4^6 (see [26]) and apply Construction 2.3 with weight 3 to obtain a $[2,1,1]$-GDC(6) of type 12^6. Here, the input $[2,1,1]$-GDC(6) of type 3^5 exists by Proposition 4.1. ■

Theorem 4.1: There exists an optimal $(12t+1,6,[2,1,1])_4$-code with size $24t^2+2t$ for all $t \geq 1$.

Proof: For all $t \geq 4$, adjoin one ideal point to a $[2,1,1]$-GDC(6) of type 12^t (Lemma 4.3) and fill in the groups together with the extra point with an optimal $(13,6,[2,1,1])_4$-code (which exists by Proposition 4.5) to obtain an optimal $(12t+1,6,[2,1,1])_4$-code with size $24t^2+2t$.

For each $t \in \{1,2,3\}$, there exists an optimal $(12t+1,6,[2,1,1])_4$-code with size $24t^2+2t$ by Proposition 4.5. ■

Theorem 4.2: There exists an optimal $(12t+7,6,[2,1,1])_4$-code with size $(12t+7)(2t+1)$ for each $1 \leq t \leq 16$.

Proof: For each $t \in \{1,2,3,4,5,6,8\}$, there exists an optimal $(12t+7,6,[2,1,1])_4$-code with size $(12t+7)(2t+1)$ by Proposition 4.5.

For the other 9 values of t, we first construct 9 GDCs of types 18^5, $24^4 18^1$, $24^4 30^1$, $30^4 18^1$, 30^5, $30^5 12^1$, $30^5 24^1$, $24^7 18^1$ and $24^7 30^1$ as follows: For $t=7$, take a $\{4\}$-GDD of type 3^5 (which exists by Lemma 2.8). Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type 18^5. For $t=9$, take a TD$(5,4)$ from Lemma 2.9. Remove one

point from a group to obtain a $\{4,5\}$-GDD of type $4^4 3^1$. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type $24^4 18^1$. For $t = 10$, take a TD$(5,5)$ from Lemma 2.9. Remove 4 points from a block to obtain a $\{4,5\}$-GDD of type $4^4 5^1$. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type $24^4 30^1$. For $t = 11$, take a TD$(5,5)$ from Lemma 2.9. Remove 2 points from a group to obtain a $\{4,5\}$-GDD of type $5^4 3^1$. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type $30^4 18^1$. For $t = 12$, take a TD$(5,5)$ from Lemma 2.9. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type 30^5. For each $t \in \{13,14\}$, take a TD$(6,5)$ from Lemma 2.9. Remove 3 or 1 points from a group to obtain a $\{4,5\}$-GDD of types $5^5 2^1$ or $5^5 4^1$. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of types $30^5 12^1$ or $30^5 24^1$. For each $t \in \{15,16\}$, take a 4-RGDD of type 4^7 (see Lemma 2.7). There are 8 parallel classes in this 4-RGDD. Add one ideal point to each of the u parallel classes for $u \in \{3,5\}$ to complete them. The result is a $\{4,5\}$-GDD of type $4^7 u^1$. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of types $24^7 18^1$ or $24^7 30^1$. Here, the input $[2,1,1]$-GDC(6)s of types 6^4 and 6^5 exist by Proposition 4.1. Adjoining one ideal point to each of the above GDCs and filling in the groups together with the extra point with an optimal code of length $n_1 \in \{13,19,25,31\}$, the result is an optimal code of length $12t+7$ with $t \in \{7,9,10,11,12,13,14,15,16\}$ as desired. ■

Theorem 4.3: There exists an optimal $(12t+7,6,[2,1,1])_4$-code with size $(12t+7)(2t+1)$ for all $t \geq 17$.

Proof: Take a TD$(6,2t)$ from Lemma 2.9. Apply Construction 2.3 with weight 6 to the points in the first 4 groups, $2x$ points in the fifth group, and y points in the last group. The other points are given weight 0. Note that there exist $[2,1,1]$-GDC(6)s of types 6^4, 6^5, 6^6 by Proposition 4.1. The result is a $[2,1,1]$-GDC(6) of type $(12t)^4(12x)^1(6y)^1$. We require that $y \geq 3$. Adjoin one ideal point and fill in the groups together with the extra point with optimal codes of lengths $12t+1$ with $t \geq 1$ (which exist by Theorem 4.1) or $12u+7$ with $1 \leq u \leq 16$ (which exist by Theorem 4.2). The result is an optimal $(48t+12x+6y+1,6,[2,1,1])_4$-code, where $48t+12x+6y+1$ can take any value greater than $12 \times 17 + 7 = 211$ except for the case of $295 = 12 \times 24 + 7$.

For $t = 24$, take a TD$(7,8)$. Apply Construction 2.3 with weight 6 to the points in the first 5 groups, 6 points in the sixth group, and 3 points in the last group. The other points are given weight 0. Note that there exist $[2,1,1]$-GDC(6)s of types 6^5, 6^6, 6^7 by Proposition 4.1. The result is a $[2,1,1]$-GDC(6) of type $48^5 36^1 18^1$. Adjoin one ideal point and fill in the groups together with the extra point with optimal codes of lengths 37, 49 (both exist by Theorem 4.1) or 19 (which exists by Theorem 4.2). The result is an optimal $(295,6,[2,1,1])_4$-code. ■

Theorem 4.4: There exists an optimal $(6t,6,[2,1,1])_4$-code with size $6t^2-3t$ for all $t \geq 1$.

Proof: The result follows by combining Lemma 4.2, Theorems 4.1–4.3 and the fact that there exists an optimal $(6,6,[2,1,1])_4$-code with size 3 (see Proposition 4.5). ■

C. The Case of Length $n \equiv 2 \pmod 6$

Lemma 4.4: Every $[2,1,1]$-GDC(6) of type 2^{3t+1} is an optimal $(6t+2,6,[2,1,1])_4$-code with size $6t^2+2t$.

Proof: The size of a $[2,1,1]$-GDC(6) of type 2^{3t+1} is $6t^2+2t$ which meets the upper bound of an optimal $(6t+2,6,[2,1,1])_4$-code. ■

Theorem 4.5: There exists an optimal $(6t+2,6,[2,1,1])_4$-code with size $6t^2+2t$ for each $t \in \{1,2,\ldots,11\} \cup \{14,17,18,22\}$.

Proof: For each $t \in \{1,2\}$, there exists an optimal $(6t+2,6,[2,1,1])_4$-code with size $6t^2+2t$ by Proposition 4.5.

For each $t \in \{3,4,5,6,7,8,9,11\}$, there exists a $[2,1,1]$-GDC(6) of type 2^{3t+1} by Proposition 4.1.

For each $t \in \{10,14,18,22\}$, take a $[2,1,1]$-GDC(6) of type $12^{t/2}$ (which exists by Lemma 4.3). Adjoin two ideal points and fill in the groups together with the two extra points with a $[2,1,1]$-GDC(6) of type $1^{12}2^1$ (which exists by Proposition 4.3). The result is an optimal $(6t+2,6,[2,1,1])_4$-code with size $6t^2+2t$.

For $t = 13$, take an optimal $(10,6,[2,1,1])_4$-code with size 10 which exists by Proposition 4.5. This code can also be regarded as a $[2,1,1]$-GDC(6) of type 1^{10}. Apply Construction 2.4 with a TD$(4,8)$ to get a $[2,1,1]$-GDC(6) of type 8^{10}. Fill in the groups with an optimal $(8,6,[2,1,1])_4$-code. The result is an optimal code of length $80 = 6 \times 13 + 2$.

For $t = 17$, take a $[2,1,1]$-GDC(6) of type 2^{13} (which exists by Proposition 4.1). Apply Construction 2.4 with a TD$(4,4)$ to get a $[2,1,1]$-GDC(6) of type 8^{13}. Fill in the groups with an optimal $(8,6,[2,1,1])_4$-code. The result is an optimal code of length $104 = 6 \times 17 + 2$. ■

Theorem 4.6: There exists an optimal $(6t+2,6,[2,1,1])_4$-code with size $6t^2+2t$ for all $t \geq 23$ and $t \in \{12,15,16,19,20,21\}$.

Proof: For each $t \geq 23$ or $t \in \{12,15,16,19,20,21\}$, take a $(t+1,\{4,5,6\},1)$-PBD from Lemma 2.4 and remove one point to obtain a $\{4,5,6\}$-GDD of type $3^i 4^j 5^k$ with $3i+4j+5k = t$. Apply construction 2.3 with weight 6 and input $[2,1,1]$-GDC(6)s of types 6^t for $t \in \{4,5,6\}$ (Lemma 4.1) to obtain a $[2,1,1]$-GDC(6) of type $18^i 24^j 30^k$ and length $6t$. Adjoining two ideal points and filling in the groups together with the two extra points with $[2,1,1]$-GDC(6)s of types 2^u for $u \in \{10,13,16\}$ (see Proposition 4.1), the result is a $[2,1,1]$-GDC(6) of type 2^{3t+1} for all $t \geq 23$ or $t \in \{12,15,16,19,20,21\}$. By Lemma 4.4, the resultant $[2,1,1]$-GDC(6) is an optimal $(6t+2,6,[2,1,1])_4$-code with size $6t^2+2t$. ■

D. The Case of Length $n \equiv 3,4 \pmod 6$

Lemma 4.5: $A_4(4,6,[2,1,1]) = 1$.

Proof: The one required codeword is $\langle 0,1,2,3 \rangle$. ■

Lemma 4.6: For any positive integer t, if $A_4(6t+4,6,[2,1,1]) = U(6t+4,6,[2,1,1])$ then $A_4(6t+3,6,[2,1,1]) = U(6t+3,6,[2,1,1])$.

Proof: In an optimal $(6t+4,6,[2,1,1])_4$-code, every coordinate has exactly $2t+1+2t+1=4t+2$ non-zero elements. Fix a coordinate x and remove all the $4t+2$ codewords containing non-zero elements in this coordinate x. Shorten all the remaining codewords by deleting the element 0 in coordinate x from them. The resultant codewords form an optimal $(6t+3,6,[2,1,1])_4$-code with size $U(6t+4,6,[2,1,1])-4t-2=6t^2+3t=U(6t+3,6,[2,1,1])$. ■

Lemma 4.7: There exists a $[2,1,1]$-GDC(6) of type 12^t9^1 for all $t\geq 4$.

Proof: For each $t\in\{4,5,\dots,15\}\cup\{17,18,19,23\}$, a $[2,1,1]$-GDC(6) of type 12^t9^1 exists by Proposition 4.2.

For $t\in\{16,21,22,26,27,28,31,32,33\}$, we first construct 9 GDCs of types 48^4, 48^460^1, 48^472^1, 48^572^1, 60^484^1, 84^4, 48^684^1, 84^448^1 and 84^460^1 as follows: For $t=16$, take a TD$(4,8)$ from Lemma 2.9. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type 48^4. For $t=21$, take a TD$(5,5)$ from Lemma 2.9. Remove 4 points from a block to obtain a $\{4,5\}$-GDD of type 4^45^1. Apply Construction 2.3 with weight 12 to obtain a $[2,1,1]$-GDC(6) of type 48^460^1. For each $t\in\{22,26\}$, take a 4-GDD of type 12^u18^1 for $u\in\{4,5\}$ from Lemma 2.8. Apply Construction 2.3 with weight 4 to obtain a $[2,1,1]$-GDC(6) of type 48^u72^1. For $t=27$, take a 4-GDD of type 15^421^1 from Lemma 2.8. Apply Construction 2.3 with weight 4 to obtain a $[2,1,1]$-GDC(6) of type 60^484^1. For $t=28$, take a TD$(4,14)$ from Lemma 2.9. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of type 84^4. For $t=31$, take a 4-GDD of type 4^67^1 from Lemma 2.8. Apply Construction 2.3 with weight 12 to obtain a $[2,1,1]$-GDC(6) of type 48^684^1. For each $t\in\{32,33\}$, take a TD$(5,14)$ from Lemma 2.9. Delete 6 or 4 points from a group to get a $\{4,5\}$-GDD of types 14^48^1 or 14^410^1. Apply Construction 2.3 with weight 6 to obtain a $[2,1,1]$-GDC(6) of types 84^448^1 or 84^460^1. Here, the input $[2,1,1]$-GDC(6)s of types $4^4,6^4,12^4$ and 12^5 exist by Proposition 4.1. Now, adjoin 9 ideal points to each of the above GDCs and fill in the groups together with the nine extra points with a $[2,1,1]$-GDC(6) of type 12^v9^1 with $v\in\{4,5,6,7\}$ (which all exist by Proposition 4.2) to obtain a $[2,1,1]$-GDC(6) of type 12^t9^1 as desired.

For all $t\geq 34$ or $t\in\{20,24,25,29,30\}$, take a $(t+1,\{5,6,7,8,9\},1)$-PBD from Lemma 2.5 and remove one point to obtain a $\{5,6,7,8,9\}$-GDD of type $4^i5^j6^k7^l8^m$ with $4i+5j+6k+7l+8m=t$. Apply construction 2.3 with weight 12 and input $[2,1,1]$-GDC(6)s of types 12^t for $t\in\{5,6,7,8,9\}$ (Lemma 4.3) to obtain a $[2,1,1]$-GDC(6) of type $48^i60^j72^k84^l96^m$ and length $12t$. Adjoin 9 ideal points and fill in the groups together with the nine extra points with $[2,1,1]$-GDC(6)s of types 12^u9^1 for $u\in\{4,5,6,7,8\}$ (which exist by Proposition 4.2) to obtain a $[2,1,1]$-GDC(6) of type 12^t9^1. ■

Theorem 4.7: There exists an optimal $(12t+10,6,[2,1,1])_4$-code with size $(4t+3)(6t+5)$ for all $t\geq 0$.

Proof: For each $t\in\{0,1,2\}$, there exists an optimal $(12t+10,6,[2,1,1])_4$-code with size $(4t+3)(6t+5)$ by Proposition 4.5.

For $t=3$, take a $[2,1,1]$-GDC(6) of type 3^5 (which exists by Proposition 4.1). Apply Construction 2.4 with a TD$(4,3)$ to get a $[2,1,1]$-GDC(6) of type 9^5. Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(10,6,[2,1,1])_4$-code. The result is an optimal code of length $46=12\times 3+10$.

For all $t\geq 4$, there exists a $[2,1,1]$-GDC(6) of type 12^t9^1. Adjoin one ideal point to this GDC and fill in the groups together with the extra point with optimal codes of lengths 10 or 13 to obtain an optimal $(12t+10,6,[2,1,1])_4$-code with size $(4t+3)(6t+5)$. ■

Theorem 4.8: There exists an optimal $(12t+16,6,[2,1,1])_4$-code with size $(4t+5)(6t+8)$ for each $0\leq t\leq 15$.

Proof: For each $t\in\{0,1\}$, there exists an optimal $(12t+16,6,[2,1,1])_4$-code with size $(4t+5)(6t+8)$ by Proposition 4.5.

For $t=2$, take a $[2,1,1]$-GDC(6) of type 10^4 (which exists by Proposition 4.1). Fill in the groups with an optimal $(10,6,[2,1,1])_4$-code. The result is an optimal code of length $40=12\times 2+16$.

For $t=3$, take a $[2,1,1]$-GDC(6) of type 13^4 (which exists by Proposition 4.1). Fill in the groups with an optimal $(13,6,[2,1,1])_4$-code. The result is an optimal code of length $52=12\times 3+16$.

For $t=4$, take a $[2,1,1]$-GDC(6) of type 3^7 (which exists by Proposition 4.1). Apply Construction 2.4 with a TD$(4,3)$ to get a $[2,1,1]$-GDC(6) of type 9^7. Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(10,6,[2,1,1])_4$-code. The result is an optimal code of length $64=12\times 4+16$.

For $t=5$, take a $[2,1,1]$-GDC(6) of type 3^5 (which exists by Proposition 4.1). Apply Construction 2.4 with a TD$(4,5)$ to get a $[2,1,1]$-GDC(6) of type 15^5. Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(16,6,[2,1,1])_4$-code. The result is an optimal code of length $76=12\times 5+16$.

For $t=6$, take a $[2,1,1]$-GDC(6) of type 22^4 (which exists by Proposition 4.1). Fill in the groups with an optimal $(22,6,[2,1,1])_4$-code. The result is an optimal code of length $88=12\times 6+16$.

For each $t\in\{7,8,\dots,15\}$, there exists a $[2,1,1]$-GDC(6) of type 12^t15^1 by Proposition 4.2. Adjoining one ideal point and filling in the groups together with the extra point with optimal codes of lengths 13 or 16, the result is an optimal $(12t+16,6,[2,1,1])_4$-code. ■

Theorem 4.9: There exists an optimal $(12t+16,6,[2,1,1])_4$-code with size $(4t+5)(6t+8)$ for all $t\geq 16$.

Proof: For all $t\geq 16$, take a TD$(6,2r)$ for $r\geq 4$ and $r\notin\{5,7,9,11\}$ from Lemma 2.9. Apply Construction 2.3 with weight 6 to the points in the first 4 groups, $2x$ points in the fifth group, and y points in the last group. We require that

$x = 0$ or $x \geq 4$ and $1 \leq y \leq 31$, y odd. The other points are given weight 0. Note that there exist $[2, 1, 1]$-GDC(6)s of types 6^4, 6^5 and 6^6 by Proposition 4.1. The result is a $[2, 1, 1]$-GDC(6) of type $(12r)^4(12x)^1(6y)^1$. Adjoining 9 ideal points and filling in the groups together with the nine extra points with $[2, 1, 1]$-GDC(6)s of types 12^u9^1 for all $u \geq 4$, the result is a $[2, 1, 1]$-GDC(6) of type $12^{4r+x}(6y + 9)^1$. Adjoin one more ideal point and fill in the groups together with the extra point with optimal codes of lengths 13 or $6y + 10$ for $1 \leq y \leq 31$ and y odd (which exist by Theorem 4.8). The result is an optimal $(48r + 12x + 6y + 10, 6, [2, 1, 1])_4$-code, where $48r + 12x + 6y + 10$ can take any value of $12t + 16$ with t greater than 16. ■

Theorem 4.10: There exists an optimal $(6t+3, 6, [2, 1, 1])_4$-code with size $6t^2 + 3t$ for all $t \geq 1$.

Proof: The result follows by combining Lemma 4.6 and Theorems 4.7–4.9. ■

E. The Case of Length $n \equiv 5 \pmod 6$

Lemma 4.8: $A_4(5, 6, [2, 1, 1]) = 1$.

Proof: The one required codeword is $\langle 0, 1, 2, 3\rangle$. ■

Theorem 4.11: There exists an optimal $(12t + 11, 6, [2, 1, 1])_4$-code with size $24t^2 + 40t + 16$ for all $t \geq 0$.

Proof: For each $t \in \{0, 1, 2\}$, there exists an optimal $(12t + 11, 6, [2, 1, 1])_4$-code with size $24t^2 + 40t + 16$ by Proposition 4.5.

For $t = 3$, there exists a $[2, 1, 1]$-GDC(6) of type $1^{36}11^1$ by Proposition 4.4. Fill in the groups with an optimal $(11, 6, [2, 1, 1])_4$-code to obtain an optimal $(47, 6, [2, 1, 1])_4$-code.

For all $t \geq 4$, there exists a $[2, 1, 1]$-GDC(6) of type 12^t9^1 by Lemma 4.7. Adjoining 2 ideal points and filling in the groups together with the two extra points with a $[2, 1, 1]$-GDC(6) of type $1^{12}2^1$ (which exists by Proposition 4.3) and an optimal $(11, 6, [2, 1, 1])_4$-code, the result is an optimal $(12t + 11, 6, [2, 1, 1])_4$-code with size $24t^2 + 40t + 16$. ■

Theorem 4.12: There exists an optimal $(12t + 17, 6, [2, 1, 1])_4$-code with size $24t^2 + 64t + 42$ for all $t \geq 0$.

Proof: For $t = 0$, there exists an optimal $(17, 6, [2, 1, 1])_4$-code with size 42 by Proposition 4.5.

For $t = 1$, take a $[2, 1, 1]$-GDC(6) of type 7^4 (which exists by Proposition 4.1). Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(8, 6, [2, 1, 1])_4$-code. The result is an optimal code of length $29 = 12 \times 1 + 17$.

For each $t \in \{2, 4, 5, 6\}$, there exists a $[2, 1, 1]$-GDC(6) of type $1^{12t+6}11^1$ by Proposition 4.4. Fill in the groups with an optimal $(11, 6, [2, 1, 1])_4$-code to obtain an optimal $(12t + 17, 6, [2, 1, 1])_4$-code.

For $t = 3$, take a $[2, 1, 1]$-GDC(6) of type 13^4 (which exists by Proposition 4.1). Adjoin one ideal point and fill in the groups together with the extra point with an optimal $(14, 6, [2, 1, 1])_4$-code. The result is an optimal code of length $53 = 12 \times 3 + 17$.

For each $t \in \{7, 8, \ldots, 15\}$, there exists a $[2, 1, 1]$-GDC(6) of type 12^t15^1 by Proposition 4.2. Adjoining 2 ideal points and filling in the groups together with the two extra points with a $[2, 1, 1]$-GDC(6) of type $1^{12}2^1$ and an optimal code of length 17, the result is an optimal $(12t+17, 6, [2, 1, 1])_4$-code.

For all $t \geq 16$, take a TD$(6, 2r)$ for $r \geq 4$ and $r \notin \{5, 7, 9, 11\}$ from Lemma 2.9. Apply Construction 2.3 with weight 6 to the points in the first 4 groups, $2x$ points in the fifth group, and y points in the last group. We require that $x = 0$ or $x \geq 4$ and $1 \leq y \leq 31$, y odd. The other points are given weight 0. Note that there exist $[2, 1, 1]$-GDC(6)s of types 6^4, 6^5 and 6^6 by Proposition 4.1. The result is a $[2, 1, 1]$-GDC(6) of type $(12r)^4(12x)^1(6y)^1$. Adjoining 9 ideal points and filling in the groups together with the nine extra points with $[2, 1, 1]$-GDC(6)s of types 12^u9^1 for all $u \geq 4$, the result is a $[2, 1, 1]$-GDC(6) of type $12^{4r+x}(6y+9)^1$. Adjoin 2 more ideal points and fill in the groups together with the two extra points with a $[2, 1, 1]$-GDC(6) of type $1^{12}2^1$ and an optimal code of length $6y+11$ with $1 \leq y \leq 31$ and y odd (which exists by Theorem 4.8). The result is an optimal $(48r+12x+6y+11, 6, [2, 1, 1])_4$-code, where $48r+12x+6y+11$ can take any value of $12t+17$ with t greater than 16. ■

V. Conclusion

In this paper, we determine almost completely the spectrum with sizes for optimal quaternary constant-composition codes with weight four and minimum distances five or six. We summarize our main results of this paper as follows:

Theorem 5.1: For any integer $n \geq 4$

$$A_4(n, 5, [2, 1, 1]) = \begin{cases} 1, & \text{if } n = 4 \\ 2, & \text{if } n = 5 \\ 6, & \text{if } n = 6 \\ 10, & \text{if } n = 7 \\ n\left\lfloor \frac{n-1}{2} \right\rfloor, & \text{if } n \geq 12 \text{ and } n \neq 13. \end{cases}$$

$$A_4(8, 5, [2, 1, 1]) \geq 18,$$

$$A_4(9, 5, [2, 1, 1]) \geq 27,$$

$$A_4(10, 5, [2, 1, 1]) \geq 36,$$

$$A_4(11, 5, [2, 1, 1]) \geq 48,$$

$$A_4(13, 5, [2, 1, 1]) \geq 72.$$

Theorem 5.2: For any integer $n \geq 4$

$$A_4(n, 6, [2, 1, 1]) = \begin{cases} 1, & \text{if } n = 4, 5 \\ 4, & \text{if } n = 7 \\ \left\lfloor \frac{n}{2} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor, & \text{if } n \geq 6 \text{ and } n \neq 7. \end{cases}$$

ACKNOWLEDGMENTS

The authors thank Prof. Olgica Milenkovic, the Associate Editor, Prof. Helmut Bölcskei, the Editor-in-Chief of IEEE Transactions on Information Theory, and the two anonymous referees for their constructive comments and suggestions that greatly improved the readability of this article.

REFERENCES

[1] R. J. R. Abel, F. E. Bennett, and M. Greig, "PBD-closure," in: *The CRC Handbook of Combinatorial Designs (Second Edition)*, C. J. Colbourn and J. H. Dinitz, Eds. Boca Raton: CRC Press, 2007, pp. 247–254.

[2] R. J. R. Abel, A. E. Brouwer, C. J. Colbourn, and J. H. Dinitz, "Mutually orthogonal Latin squares (MOLS)," in *The CRC Handbook of Combinatorial Designs (Second Edition)*, C. J. Colbourn and J. H. Dinitz, Eds. Boca Raton: CRC Press, 2007, pp. 160–193.

[3] G. T. Bogdanova, "Bounds for the Maximum Size of Ternary Constant-Composition Codes", in *Proc. of the International Workshop on Optimal Codes*, Jun. 1998, pp. 15–18.

[4] G. T. Bogdanova and S. N. Kapralov, "Enumeration of optimal ternary codes with a given composition," *Problemy Peredachi Informatsii*, vol. 39, no. 4, pp. 35–40, 2003.

[5] G.T. Bogdanova and D.S Ocetarova, "Some Ternary Constant-Composition Codes", in *Proc. Sixth Int. Workshop Algebraic and Combinatorial Coding Theory*, Pskov, Russia. Sep. 1998, pp. 41–45,

[6] Y. M. Chee, S. H. Dau, A. C. H. Ling, and S. Ling, "Linear size optimal q-ary constant-weight codes and constant-composition codes" *IEEE Trans. Inform. Theory*, vol. 56 , no. 1, pp. 140–151, 2010.

[7] Y. M. Chee, G. Ge, and A. C. H. Ling, "Group divisible codes and their application in the construction of optimal constnant-composition codes of weight three," *IEEE Trans. Inform. Theory*, vol. 54, no. 8, pp. 3552–3564, 2008.

[8] Y. M. Chee, A. C. H. Ling, S. Ling, and H. Shen, "The PBD-closure of constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 53, no. 8, pp. 2685–2692, 2007.

[9] Y. M. Chee and S. Ling, "Improved lower bounds for constant GC-content DNA codes," *IEEE Trans. Inform. Theory*, vol. 54, no. 1, pp. 391–394, 2008.

[10] W. Chu, C. J. Colbourn, and P. Dukes, "Constructions for permutation codes in powerline communications," *Des. Codes Cryptogr.*, vol. 32, no. 1-3, pp. 51–64, 2004.

[11] ——, "On constant composition codes," *Discrete Appl. Math.*, vol. 154, no. 6, pp. 912–929, 2006.

[12] C. J. Colbourn, T. Klϕve, and A. C. H. Ling, "Permutation arrays for powerline communication and mutually orthogonal Latin squares," *IEEE Trans. Inform. Theory*, vol. 50, no. 6, pp. 1289–1291, 2004.

[13] D. J. Costello and G. D. Formey, "Channel coding: The road to channel capacity," *Proc. IEEE*, vol. 95, no. 6, pp. 1150–1177, 2007.

[14] A. G. D'yachkov, "Random constant composition codes for multiple access channels," *Problems Control Inform. Theory/Problemy Upravlen. Teor. Inform.*, vol. 13, no. 6, pp. 357–369, 1984.

[15] C. Ding, "Optimal Constant Composition Codes From Zero-Difference Balanced Functions," *IEEE Trans. Inform. Theory*, vol. 54, no. 12, pp. 5766–5770, 2008.

[16] C. Ding and J. Yin, "Algebraic constrctions of constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 4, pp. 1585–1589, 2005.

[17] ——, "Combinatorial constructions of optimal constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 10, pp. 3671–3674, 2005.

[18] ——, "A construction of optimal constant composition codes," *Des. Codes Cryptogr.*, vol. 40, no. 2, pp. 157–165, 2006.

[19] C. Ding and J. Yuan, "A family of optimal constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 10, pp. 3668–3671, 2005.

[20] Y. Ding, "A construction for constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 54, no. 10, pp. 3738–3741, 2008.

[21] J. H. Dinitz, Room squares, in: *The CRC Handbook of Combinatorial Designs (Second Edition)*, C. J. Colbourn and J. H. Dinitz, Eds. Boca Raton: CRC Press, 2007, pp. 584–590.

[22] J. H. Dinitz and D. R. Stinson, "The construction and uses of frames" *Ars Combin.*, vol. 10, pp. 31–53, 1980.

[23] T. Ericson and V. Zinoview, "Spherical codes generated by binary parititions of symmetric pointsets," *IEEE Trans. Inform. Theory*, vol. 41, no. 1, pp. 107–129, 1995.

[24] F. Gao and G. Ge, "Optimal ternary constant composition codes of weight four and distance five," *IEEE Trans. Inform. Theory*, vol.57, no. 6, pp. 3742–3757, 2011.

[25] G. Ge, Group divisible designs, in: *The CRC Handbook of Combinatorial Designs (Second Edition)*, C. J. Colbourn and J. H. Dinitz, Eds. Boca Raton: CRC Press, 2007, pp. 255–260.

[26] G. Ge and A. C. H. Ling, "Asymptotic results on the existence of 4-RGDDs and uniform 5-GDDs," *J. Combin. Des.*, vol. 13, no. 3, pp. 222–237, 2005.

[27] S. Huczynska, "Equidistant frequency permutation arrays and related constant composition codes," *Des. Codes Cryptogr.*, vol. 54, no. 2, pp. 109–120, 2010.

[28] S. Huczynska and G. L. Mullen, "Frequency permutation arrays," *J. Combin. Des.*, vol. 14, no. 6, pp. 463–478, 2006.

[29] O. D. King, "Bounds for DNA codes with constant GC-content," *Electron. J. Combin.*, vol. 10, no. 1, 2003, Research Paper 33 (electronic).

[30] A. C. H. Ling, X. Zhu, C. J. Colbourn, and R. C. Mullin, "Pairwise balanced designs with consecutive block sizes," *Des. Codes Cryptogr.*, vol. 10, no. 2, pp. 203–222, 1997.

[31] Y. Luo, F. W. Fu, A. J. H. Vinck, and W. Chen, "On constant-composition codes over $\mathbb{Z}_q$," *IEEE Trans. Inform. Theory*, vol. 49, no. 11, pp. 3010–3016, 2003.

[32] O. Milenkovic and N. Kashyap, *On the design of codes for DNA computing (Lecture Notes in Computer Science)*. Berlin, Germany: Springer-Verlag, 2006, vol. 3969, pp. 100–119.

[33] R. Rees and D. R. Stinson, "On the existence of incomplete designs of block size four having one hole," *Utilitas Math.*, vol. 35, pp. 119–152, 1989.

[34] D. R. Stinson, "The spectrum of skew Room squares," *J. Austral. Math. Soc. Ser. A* , vol. 31 (4), pp. 475–480, 1981.

[35] M. Svanström, "Ternary codes with weight constraints," Ph.D. dissertation. Linköpings Universitet, Sweden, 1999.

[36] ——, "Constructions of ternary constant-composition codes with weight three," *IEEE Trans. Inform. Theory*, vol. 46, no. 7, pp. 2644–2647, 2000.

[37] M. Svanström, P. R. J. Östergård, and G. T. Bogdanova, "Bounds and constructions for ternary constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 48, no. 1, pp. 101–111, 2002.

[38] I. E. Telatar and R. G. Gallager, "Zero error decision feedback capacity of discrete memoryless channels," in *Proc. Bilkent Int. Conf. New Trends Commun. Control Signal Process.*, E. Arikan, Ed., 1990, pp. 228–233, Elsevier.

[39] B. Wen, J. Wang, and J. Yin, "Optimal grid holey packings with block size 3 and 4," *Des. Codes Cryptogr.*, vol. 52, no. 1, pp. 107–124, 2009.

[40] J. Yan and J. Yin, "Constructions of optimal GDRP$(n, \lambda; v)$'s of type $\lambda^1\mu^{m-1}$," *Discrete Appl. Math.*, vol. 156, no. 14, pp. 2666–2678, 2008.

[41] ——, "A class of optimal constant composition codes from GDRPs," *Des. Codes Cryptogr.*, vol. 50, no. 1, pp. 61–76, 2009.

[42] J. Yin and Y. Tang, "A new combinatorial approach to the construction of constant composition codes," *Sci. China Ser. A*, vol. 51, no. 3, pp. 416–426, 2008.

[43] J. Yin, J. Yan, and C. Wang, "Generalized balanced tournament designs and related codes," *Des. Codes Cryptogr.*, vol. 46, no. 2, pp. 211–230, 2008.

The following information is for referees only, not for publication.

VI. Base codewords for CCCs and GDCs with distance 5 and type $[2,1,1]$

Proposition 6.1: There exists a $[2,1,1]$-GDC(5) of type 2^t with size $2t(t-1)$ for each $t \in \{9,11\}$, which is also an optimal $(2t,5,[2,1,1])_4$-code.

Proof: For each $t \in \{9,11\}$, let $X_t = \mathbb{Z}_{2t}$, $\mathcal{G}_t = \{\{i, t+i\} : i \in \mathbb{Z}_t\}$ and $\mathcal{C}_t$ be the set of cyclic (or quasi-cyclic) shifts of the vectors generated by the following vectors respectively. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2,1,1]$-GDC(5) of type 2^t with size $2t(t-1)$, where

1) $t=9$, $n=18$, $m=5$, $s=2$, $M=1$

$$\begin{array}{llll} P: & \langle 0,2,1,15\rangle & & \\ R: & \langle 0,3,15,17\rangle & \langle 0,11,14,1\rangle & \langle 0,13,6,7\rangle \\ & \langle 0,12,4,10\rangle & \langle 0,1,8,5\rangle & \langle 0,4,2,6\rangle \end{array}$$

2) $t=11$, $n=22$, $m=3$, $s=5$, $M=2$

$$\begin{array}{llll} P: & \langle 0,1,2,3\rangle & \langle 0,2,7,4\rangle & \langle 0,7,20,19\rangle \\ & \langle 1,3,5,8\rangle. & & \end{array}$$

■

Proposition 6.2: There exists a $[2,1,1]$-GDC(5) of type 3^t with size $\frac{9t(t-1)}{2}$ for each $t \in \{7,9\}$.

Proof: For each $t \in \{7,9\}$, let $X_t = \mathbb{Z}_{3t}$, $\mathcal{G}_t = \{\{i, t+i, 2t+i\} : i \in \mathbb{Z}_t\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2,1,1]$-GDC(5) of type 3^t with size $\frac{9t(t-1)}{2}$, where

1) $t=7$, $n=21$, $m=5$, $s=2$, $M=1$

$$\begin{array}{llll} P: & \langle 0,13,16,18\rangle & \langle 0,12,4,8\rangle & \\ R: & \langle 0,10,8,2\rangle & \langle 0,17,1,12\rangle & \langle 0,15,12,3\rangle \\ & \langle 0,1,10,11\rangle & \langle 0,5,11,20\rangle & \end{array}$$

2) $t=9$, $n=27$, $m=2$, $s=3$, $M=1$

$$\begin{array}{llll} P: & \langle 0,1,23,6\rangle & \langle 0,8,12,7\rangle & \\ R: & \langle 0,14,15,17\rangle & \langle 0,21,14,2\rangle & \langle 0,17,3,11\rangle \\ & \langle 0,20,26,15\rangle & \langle 0,15,25,19\rangle & \langle 0,3,5,16\rangle. \end{array}$$

■

Proposition 6.3: There exists a $[2,1,1]$-GDC(5) of type 4^t with size $8t(t-1)$ for each $t \in \{5,6,7,8,9,11\}$.

Proof: For each $t \in \{5,6,7,8,9,11\}$, let $X_t = \mathbb{Z}_{4t}$, $\mathcal{G}_t = \{\{i, t+i, 2t+i, 3t+i\} : i \in \mathbb{Z}_t\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2,1,1]$-GDC(5) of type 4^t with size $8t(t-1)$, where

1) $t=5$, $n=20$, $m=3$, $s=2$, $M=1$

$$\begin{array}{llll} P: & \langle 0,4,11,13\rangle & \langle 0,6,4,2\rangle & \\ R: & \langle 0,9,6,3\rangle & \langle 0,1,9,12\rangle & \langle 0,3,2,1\rangle \\ & \langle 0,7,3,4\rangle & & \end{array}$$

2) $t=6$, $n=24$, $m=29$, $s=2$, $M=1$

$$\begin{array}{llll} P: & \langle 0,11,9,10\rangle & \langle 0,10,23,21\rangle & \\ R: & \langle 0,15,16,8\rangle & \langle 0,1,4,15\rangle & \langle 0,20,7,16\rangle \\ & \langle 0,3,8,4\rangle & \langle 0,5,20,3\rangle & \langle 0,8,10,13\rangle \end{array}$$

3) $t=7$, $n=28$, $m=3$, $s=3$, $M=1$

$$\begin{array}{llll} P: & \langle 0,8,5,4\rangle & \langle 0,13,23,15\rangle & \langle 0,27,3,25\rangle \\ R: & \langle 0,2,22,11\rangle & \langle 0,6,24,5\rangle & \langle 0,10,26,13\rangle \end{array}$$

4) $t=8$, $n=32$, $m=5$, $s=3$, $M=1$

$$\begin{array}{llll} P: & \langle 0,3,14,21\rangle & \langle 0,1,27,6\rangle & \\ R: & \langle 0,19,12,7\rangle & \langle 0,4,1,19\rangle & \langle 0,22,5,17\rangle \\ & \langle 0,14,31,28\rangle & \langle 0,20,9,11\rangle & \langle 0,2,20,1\rangle \\ & \langle 0,6,28,10\rangle & \langle 0,9,13,12\rangle & \end{array}$$

5) $t=9$, $n=36$, $m=11$, $s=2$, $M=1$

$$\begin{array}{llll} P: & \langle 0,5,21,19\rangle & \langle 0,32,30,7\rangle & \langle 0,13,33,2\rangle \\ & \langle 0,14,24,4\rangle & & \\ R: & \langle 0,3,31,6\rangle & \langle 0,34,35,30\rangle & \langle 0,21,26,20\rangle \\ & \langle 0,24,13,16\rangle & \langle 0,6,17,21\rangle & \langle 0,7,29,31\rangle \\ & \langle 0,11,19,12\rangle & \langle 0,16,23,33\rangle & \end{array}$$

6) $t=11$, $n=44$, $m=3$, $s=5$, $M=1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,8,16,25\rangle & \langle 0,18,41,14\rangle \\ & \langle 0,21,19,39\rangle. & & \end{array}$$

■

Proposition 6.4: There exists a $[2,1,1]$-GDC(5) of type 6^5 with size 360.

Proof: Let $X = \mathbb{Z}_{30}$, $\mathcal{G} = \{\{i, 5+i, 10+i, 15+i, 20+i, 25+i\} : i \in \mathbb{Z}_5\}$ and $\mathcal{C}$ be the set of cyclic shifts of the vectors generated by the following vectors. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[2,1,1]$-GDC(5) of type 6^5 with size 360, where $n=30$, $m=13$, $s=2$, $M=1$ and

$$\begin{array}{llll} P: & \langle 0,9,7,3\rangle & \langle 0,2,29,28\rangle & \\ R: & \langle 0,18,26,29\rangle & \langle 0,13,22,19\rangle & \langle 0,11,14,2\rangle \\ & \langle 0,14,16,27\rangle & \langle 0,7,13,14\rangle & \langle 0,22,11,23\rangle \\ & \langle 0,29,23,17\rangle & \langle 0,6,18,22\rangle. & \end{array}$$

■

Proposition 6.5: There exists a $[2,1,1]$-GDC(5) of type 4^t2^1 with size $8t^2$ for each $t \in \{5,6\}$.

Proof: For each $t \in \{5,6\}$, let $X_t = \mathbb{Z}_{4t} \cup \{a,b\}$, $\mathcal{G}_t = \{\{i, t+i, 2t+i, 3t+i\} : i \in \mathbb{Z}_t\} \cup \{\{a,b\}\}$ and $\mathcal{C}_t$ be the set of quasi-cyclic shifts of the vectors generated by the following vectors respectively. Here, the elements a, b keep fixed under the action of the automorphism group. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2,1,1]$-GDC(5) of type 4^t2^1 with size $8t^2$, where

1) $t=5$, $m=3$, $s=2$, $M=2$

$$\begin{array}{llll} P: & \langle 1,19,13,7\rangle & \langle 1,5,8,14\rangle & \langle 0,4,1,7\rangle \\ & \langle 0,2,14,8\rangle & & \\ R: & \langle 0,17,a,11\rangle & \langle 1,12,5,4\rangle & \langle 0,19,7,16\rangle \\ & \langle 1,18,7,a\rangle & \langle 1,14,0,b\rangle & \langle 1,8,b,2\rangle \\ & \langle b,0,8,19\rangle & \langle 0,1,18,17\rangle & \langle b,1,19,12\rangle \\ & \langle a,0,19,2\rangle & \langle a,1,12,13\rangle & \langle 0,11,4,13\rangle \end{array}$$

2) $t = 6$, $m = 5$, $s = 2$, $M = 2$

$$\begin{array}{llll} P: & \langle 0,23,10,14\rangle & \langle 0,13,8,11\rangle & \langle 0,22,7,15\rangle \\ & \langle 0,5,15,10\rangle & & \\ R: & \langle 0,7,4,23\rangle & \langle a,0,20,21\rangle & \langle 0,20,1,4\rangle \\ & \langle 1,23,14,3\rangle & \langle 1,14,9,10\rangle & \langle 1,17,21,12\rangle \\ & \langle b,0,23,19\rangle & \langle 1,22,15,b\rangle & \langle 0,8,22,9\rangle \\ & \langle 1,10,23,a\rangle & \langle 1,11,4,21\rangle & \langle a,1,17,14\rangle \\ & \langle b,1,10,0\rangle & \langle 0,21,b,5\rangle & \langle 0,9,a,16\rangle \\ & \langle 1,5,6,22\rangle. & & \end{array}$$

■

Proposition 6.6: $A_4(n,5,[2,1,1]) = U(n,5,[2,1,1])$ for each $n \in \{12,14,15,17,20,28,30,36,38,44,46,52,54,62,68,70,76,78,86,92,94,110,126,134\}$.

Proof: For each given n, we take point set $X = \mathbb{Z}_n$. The codes are the sets of cyclic shifts of the vectors generated by the following vectors respectively.

1) $n = 12$, $m = 1$, $s = 1$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,3,9\rangle & \langle 0,2,7,5\rangle & \langle 0,3,11,7\rangle \\ & \langle 0,4,10,2\rangle & \langle 0,5,9,11\rangle & \end{array}$$

2) $n = 14$, $m = 9$, $s = 3$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,10\rangle & & \\ R: & \langle 0,4,10,7\rangle & \langle 0,6,13,5\rangle & \langle 0,2,5,4\rangle \end{array}$$

3) $n = 15$, $m = 1$, $s = 1$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,12,9\rangle & \langle 0,3,9,13\rangle \\ & \langle 0,4,7,6\rangle & \langle 0,5,13,1\rangle & \langle 0,6,5,14\rangle \\ & \langle 0,7,11,12\rangle & & \end{array}$$

4) $n = 17$, $m = 2$, $s = 2$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,7,6\rangle & & \\ R: & \langle 0,6,2,9\rangle & \langle 0,8,11,15\rangle & \langle 0,3,1,2\rangle \\ & \langle 0,10,9,4\rangle & \langle 0,4,8,1\rangle & \langle 0,5,10,13\rangle \end{array}$$

5) $n = 20$, $m = 3$, $s = 2$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,18,9,16\rangle & \langle 0,4,12,17\rangle & \langle 0,3,5,4\rangle \\ R: & \langle 0,7,10,2\rangle & \langle 0,1,18,7\rangle & \langle 0,5,19,10\rangle \end{array}$$

6) $n = 28$, $m = 5$, $s = 2$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,11,8\rangle & \langle 0,3,18,16\rangle \\ R: & \langle 0,21,24,19\rangle & \langle 0,16,20,21\rangle & \langle 0,4,26,18\rangle \\ & \langle 0,6,14,23\rangle & \langle 0,20,13,27\rangle & \langle 0,9,16,10\rangle \\ & \langle 0,11,23,22\rangle & & \end{array}$$

7) $n = 30$, $m = 23$, $s = 3$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,11,14\rangle & \\ R: & \langle 0,25,12,24\rangle & \langle 0,26,6,7\rangle & \langle 0,9,5,19\rangle \\ & \langle 0,12,7,13\rangle & \langle 0,24,22,17\rangle & \langle 0,13,3,21\rangle \\ & \langle 0,27,15,25\rangle & \langle 0,10,24,15\rangle & \end{array}$$

8) $n = 36$, $m = 31$, $s = 2$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,5,9\rangle & \langle 0,3,13,14\rangle \\ & \langle 0,4,27,23\rangle & \langle 0,11,15,8\rangle & \\ R: & \langle 0,22,18,10\rangle & \langle 0,9,34,31\rangle & \langle 0,23,35,12\rangle \\ & \langle 0,29,17,28\rangle & \langle 0,8,28,6\rangle & \langle 0,6,14,26\rangle \\ & \langle 0,12,6,30\rangle & & \end{array}$$

9) $n = 38$, $m = 13$, $s = 4$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,32,26,21\rangle & \langle 0,30,5,14\rangle & \langle 0,7,37,13\rangle \\ R: & \langle 0,14,15,11\rangle & \langle 0,21,35,29\rangle & \langle 0,29,2,33\rangle \\ & \langle 0,37,28,18\rangle & \langle 0,25,20,37\rangle & \langle 0,3,22,28\rangle \end{array}$$

10) $n = 44$, $m = 5$, $s = 4$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,14,7\rangle & \langle 0,3,24,21\rangle \\ R: & \langle 0,24,33,32\rangle & \langle 0,36,39,28\rangle & \langle 0,27,23,40\rangle \\ & \langle 0,16,35,14\rangle & \langle 0,33,7,11\rangle & \langle 0,18,38,1\rangle \\ & \langle 0,4,15,34\rangle & \langle 0,9,22,33\rangle & \langle 0,12,43,38\rangle \end{array}$$

11) $n = 46$, $m = 7$, $s = 4$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,12,20\rangle & \langle 0,5,32,29\rangle \\ R: & \langle 0,30,41,31\rangle & \langle 0,17,34,16\rangle & \langle 0,8,16,43\rangle \\ & \langle 0,26,13,33\rangle & \langle 0,19,44,13\rangle & \langle 0,22,45,15\rangle \\ & \langle 0,34,31,25\rangle & \langle 0,9,18,36\rangle & \langle 0,36,29,22\rangle \\ & \langle 0,18,37,23\rangle & & \end{array}$$

12) $n = 52$, $m = 7$, $s = 5$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,5,24\rangle & \langle 0,4,20,5\rangle \\ R: & \langle 0,44,30,2\rangle & \langle 0,13,19,26\rangle & \langle 0,25,12,44\rangle \\ & \langle 0,17,32,11\rangle & \langle 0,37,11,15\rangle & \langle 0,22,47,17\rangle \\ & \langle 0,5,34,39\rangle & \langle 0,41,22,18\rangle & \langle 0,19,17,33\rangle \\ & \langle 0,23,13,48\rangle & & \end{array}$$

13) $n = 54$, $m = 29$, $s = 4$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,5,12\rangle & \langle 0,5,11,1\rangle \\ & \langle 0,10,32,53\rangle & & \\ R: & \langle 0,45,43,6\rangle & \langle 0,12,53,30\rangle & \langle 0,32,23,34\rangle \\ & \langle 0,51,18,11\rangle & \langle 0,36,28,27\rangle & \langle 0,24,50,41\rangle \\ & \langle 0,48,19,7\rangle & \langle 0,11,38,9\rangle & \langle 0,33,42,28\rangle \\ & \langle 0,15,30,36\rangle & & \end{array}$$

14) $n = 62$, $m = 7$, $s = 7$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,17,42\rangle & \langle 0,3,59,51\rangle \\ R: & \langle 0,22,46,39\rangle & \langle 0,32,22,61\rangle & \langle 0,46,44,59\rangle \\ & \langle 0,42,11,5\rangle & \langle 0,38,34,31\rangle & \langle 0,50,42,45\rangle \\ & \langle 0,53,9,41\rangle & \langle 0,18,48,53\rangle & \langle 0,6,12,33\rangle \end{array}$$

15) $n = 68$, $m = 7$, $s = 7$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,5,64,19\rangle & \langle 0,2,29,66\rangle \\ R: & \langle 0,58,62,51\rangle & \langle 0,40,10,25\rangle & \langle 0,4,58,17\rangle \\ & \langle 0,60,28,41\rangle & \langle 0,24,50,1\rangle & \langle 0,36,12,2\rangle \\ & \langle 0,51,34,20\rangle & \langle 0,16,33,15\rangle & \langle 0,12,60,43\rangle \\ & \langle 0,20,66,7\rangle & \langle 0,25,20,48\rangle & \langle 0,29,45,39\rangle \end{array}$$

16) $n = 70$, $m = 13$, $s = 4$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,6,9\rangle & \langle 0,3,12,16\rangle \\ & \langle 0,4,3,22\rangle & \langle 0,6,37,64\rangle & \\ R: & \langle 0,56,18,61\rangle & \langle 0,61,36,28\rangle & \langle 0,51,5,23\rangle \\ & \langle 0,50,10,25\rangle & \langle 0,60,50,31\rangle & \langle 0,47,25,57\rangle \\ & \langle 0,45,66,40\rangle & \langle 0,40,35,59\rangle & \langle 0,63,55,53\rangle \\ & \langle 0,49,42,35\rangle & \langle 0,65,15,50\rangle & \langle 0,28,56,43\rangle \\ & \langle 0,15,64,14\rangle & \langle 0,33,40,30\rangle & \end{array}$$

17) $n = 76$, $m = 25$, $s = 7$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,2,6,7\rangle & \langle 0,4,7,10\rangle \\ & \langle 0,7,40,48\rangle & & \\ R: & \langle 0,9,18,38\rangle & \langle 0,73,70,64\rangle & \langle 0,63,56,52\rangle \\ & \langle 0,58,36,45\rangle & \langle 0,6,64,61\rangle & \langle 0,21,53,57\rangle \\ & \langle 0,57,19,8\rangle & \langle 0,64,55,46\rangle & \langle 0,36,63,14\rangle \end{array}$$

18) $n = 78$, $m = 11$, $s = 5$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,17,76,34\rangle & \langle 0,9,15,56\rangle & \langle 0,35,42,2\rangle \\ & \langle 0,14,63,18\rangle & \langle 0,18,28,73\rangle & \\ R: & \langle 0,32,5,75\rangle & \langle 0,13,29,9\rangle & \langle 0,40,26,66\rangle \\ & \langle 0,41,60,30\rangle & \langle 0,30,44,3\rangle & \langle 0,58,2,15\rangle \\ & \langle 0,50,27,24\rangle & \langle 0,62,37,5\rangle & \langle 0,11,31,39\rangle \\ & \langle 0,26,65,13\rangle & \langle 0,15,38,48\rangle & \langle 0,44,52,50\rangle \\ & \langle 0,4,17,40\rangle & & \end{array}$$

19) $n = 86$, $m = 9$, $s = 7$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,3,15,25\rangle & \langle 0,1,2,37\rangle & \langle 0,6,16,11\rangle \\ & \langle 0,2,5,74\rangle & & \\ R: & \langle 0,72,70,68\rangle & \langle 0,28,51,77\rangle & \langle 0,57,85,27\rangle \\ & \langle 0,63,29,6\rangle & \langle 0,52,35,51\rangle & \langle 0,16,46,59\rangle \\ & \langle 0,17,77,12\rangle & \langle 0,42,24,10\rangle & \langle 0,67,38,76\rangle \\ & \langle 0,40,83,4\rangle & \langle 0,48,67,3\rangle & \langle 0,65,59,83\rangle \\ & \langle 0,7,72,15\rangle & \langle 0,35,79,20\rangle & \end{array}$$

20) $n = 92$, $m = 7$, $s = 11$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,1,2,4\rangle & \langle 0,16,8,13\rangle & \langle 0,2,47,90\rangle \\ R: & \langle 0,21,90,51\rangle & \langle 0,47,38,9\rangle & \langle 0,35,46,69\rangle \\ & \langle 0,75,62,6\rangle & \langle 0,33,74,14\rangle & \langle 0,23,66,1\rangle \\ & \langle 0,3,25,49\rangle & \langle 0,27,85,2\rangle & \langle 0,61,19,79\rangle \\ & \langle 0,39,29,81\rangle & \langle 0,5,91,15\rangle & \langle 0,37,23,63\rangle \end{array}$$

21) $n = 94$, $m = 5$, $s = 11$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,64,45,75\rangle & \langle 0,21,14,9\rangle & \langle 0,80,24,64\rangle \\ R: & \langle 0,31,66,67\rangle & \langle 0,23,78,26\rangle & \langle 0,88,15,23\rangle \\ & \langle 0,89,67,81\rangle & \langle 0,61,64,47\rangle & \langle 0,1,30,53\rangle \\ & \langle 0,15,9,66\rangle & \langle 0,19,51,54\rangle & \langle 0,9,53,57\rangle \\ & \langle 0,3,84,24\rangle & \langle 0,25,77,74\rangle & \langle 0,45,92,77\rangle \\ & \langle 0,37,48,15\rangle & & \end{array}$$

22) $n = 110$, $m = 57$, $s = 9$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,38,25,2\rangle & \langle 0,17,1,85\rangle & \langle 0,14,26,13\rangle \\ & \langle 0,69,10,59\rangle & & \\ R: & \langle 0,60,66,106\rangle & \langle 0,100,7,82\rangle & \langle 0,15,68,44\rangle \\ & \langle 0,105,11,50\rangle & \langle 0,37,55,15\rangle & \langle 0,74,36,108\rangle \\ & \langle 0,25,83,103\rangle & \langle 0,20,33,97\rangle & \langle 0,65,32,48\rangle \\ & \langle 0,22,44,11\rangle & \langle 0,70,49,54\rangle & \langle 0,80,4,27\rangle \\ & \langle 0,99,107,102\rangle & \langle 0,44,43,1\rangle & \langle 0,19,88,41\rangle \\ & \langle 0,75,60,21\rangle & \langle 0,18,99,79\rangle & \langle 0,33,64,66\rangle \end{array}$$

23) $n = 126$, $m = 11$, $s = 6$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,50,87,20\rangle & \langle 0,69,58,10\rangle & \langle 0,113,52,2\rangle \\ & \langle 0,52,100,35\rangle & \langle 0,96,41,11\rangle & \langle 0,106,78,44\rangle \\ & \langle 0,103,110,75\rangle & \langle 0,37,57,68\rangle & \\ R: & \langle 0,108,99,42\rangle & \langle 0,7,90,108\rangle & \langle 0,99,89,78\rangle \\ & \langle 0,49,108,54\rangle & \langle 0,81,97,99\rangle & \langle 0,84,21,72\rangle \\ & \langle 0,98,18,125\rangle & \langle 0,35,54,90\rangle & \langle 0,70,72,115\rangle \\ & \langle 0,14,50,117\rangle & \langle 0,90,9,84\rangle & \langle 0,117,22,21\rangle \\ & \langle 0,21,105,102\rangle & \langle 0,54,81,63\rangle & \end{array}$$

24) $n = 134$, $m = 117$, $s = 10$, $M = 1$

$$\begin{array}{llll} P: & \langle 0,5,81,30\rangle & \langle 0,120,32,132\rangle & \langle 0,79,24,74\rangle \\ & \langle 0,23,63,62\rangle & \langle 0,72,25,9\rangle & \\ R: & \langle 0,87,74,101\rangle & \langle 0,112,84,75\rangle & \langle 0,56,61,67\rangle \\ & \langle 0,118,75,20\rangle & \langle 0,66,104,93\rangle & \langle 0,82,49,54\rangle \\ & \langle 0,50,35,131\rangle & \langle 0,13,19,78\rangle & \langle 0,90,72,8\rangle \\ & \langle 0,54,65,112\rangle & \langle 0,60,83,79\rangle & \langle 0,46,10,77\rangle \\ & \langle 0,114,94,68\rangle & \langle 0,106,80,23\rangle & \langle 0,4,109,90\rangle \\ & \langle 0,15,82,50\rangle. & & \end{array}$$

■

Proposition 6.7: $A_4(n,5,[2,1,1]) = U(n,5,[2,1,1])$ for each $n \in \{21,23,24,25,26,27,29,31,32,33,34,35,37,39,40,41,42,43,45,47,48,49,50,51,53,55,56,57,58,59,61,63,64,65,66,67,69,71,73,74,75,77,79,83,85,87,88,89,93,95,99,103,104,106,107,109,111,123,125,127,131,133,138,139\}$.

Proof: All these optimal codes are constructed by strong starters or strong frame starters with n odd or even respectively. The starters are given in a similar way as the codewords in the above propositions.

1) $n = 21$, $m = 1$, $s = 1$, $M = 1$

$$\begin{array}{ll} P: & \{3,19\},\{2,14\},\{6,13\},\{8,10\},\{15,16\}, \\ & \{11,17\},\{5,9\},\{1,4\},\{7,18\},\{12,20\} \end{array}$$

2) $n = 23$, $m = 2$, $s = 11$, $M = 1$

$$P: \quad \{1,5\}$$

3) $n = 24$, $m = 5$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{1,10\},\{8,21\} \\ R: & \{3,7\},\{4,18\},\{6,11\},\{13,15\},\{14,22\}, \\ & \{17,23\},\{19,20\} \end{array}$$

4) $n = 25$, $m = 2$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{1,8\},\{6,9\},\{7,15\},\{10,23\} \\ R: & \{3,24\},\{4,19\},\{11,13\},\{17,22\} \end{array}$$

5) $n = 26$, $m = 3$, $s = 3$, $M = 1$

$$P: \quad \{1,5\},\{2,10\},\{7,22\},\{8,25\}$$

6) $n = 27$, $m = 5$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{13,17\},\{7,22\},\{6,14\} \\ R: & \{1,10\},\{5,21\},\{9,12\},\{15,25\},\{18,23\}, \\ & \{19,20\},\{24,26\} \end{array}$$

7) $n = 29$, $m = 7$, $s = 7$, $M = 1$

$$P: \quad \{1,3\},\{4,10\}$$

8) $n = 31$, $m = 7$, $s = 15$, $M = 1$

$$P: \quad \{1,3\}$$

9) $n = 32$, $m = 3$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{13,18\},\{19,30\},\{3,31\},\{11,14\},\{8,27\}, \\ & \{2,12\} \\ R: & \{5,23\},\{15,21\},\{20,28\} \end{array}$$

10) $n = 33$, $m = 5$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{1,15\},\{6,11\},\{16,29\},\{17,24\},\{7,31\}, \\ & \{4,27\},\{25,28\} \\ R: & \{10,32\},\{12,18\} \end{array}$$

11) $n = 34$, $m = 3$, $s = 4$, $M = 1$

$$P: \quad \{1,4\},\{13,18\},\{16,30\},\{21,33\}$$

12) $n = 35$, $m = 2$, $s = 3$, $M = 1$

$$\begin{array}{ll} P: & \{1,3\},\{7,20\},\{22,25\},\{29,34\} \\ R: & \{8,24\},\{13,27\},\{16,17\},\{19,26\},\{21,32\} \end{array}$$

13) $n = 37$, $m = 7$, $s = 9$, $M = 1$

$$P: \quad \{1,3\},\{2,5\}$$

14) $n = 39$, $m = 4$, $s = 3$, $M = 1$

$$\begin{array}{ll} P: & \{11,34\},\{2,31\},\{21,27\},\{18,29\} \\ R: & \{1,4\},\{9,16\},\{10,12\},\{13,22\},\{14,26\}, \\ & \{17,25\},\{23,36\} \end{array}$$

15) $n = 40$, $m = 3$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{1,18\},\{4,7\},\{9,36\},\{17,35\},\{22,24\}, \\ & \{16,23\},\{31,39\},\{2,30\},\{33,38\} \\ R: & \{5,15\} \end{array}$$

16) $n = 41$, $m = 10$, $s = 5$, $M = 1$

$$P: \quad \{1,3\},\{2,5\},\{4,14\},\{11,35\}$$

17) $n = 42$, $m = 23$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{1,18\},\{5,38\},\{4,6\},\{22,27\},\{7,25\}, \\ & \{9,41\},\{16,17\},\{20,28\} \\ R: & \{3,15\},\{10,24\},\{11,26\},\{30,37\} \end{array}$$

18) $n = 43$, $m = 9$, $s = 21$, $M = 1$

$$P: \quad \{1,3\}$$

19) $n = 45$, $m = 2$, $s = 3$, $M = 1$

$$\begin{array}{ll} P: & \{13,23\},\{4,28\},\{18,31\},\{6,20\},\{21,44\} \\ R: & \{3,15\},\{5,32\},\{9,25\},\{10,19\},\{14,29\}, \\ & \{30,38\},\{33,37\} \end{array}$$

20) $n = 47$, $m = 2$, $s = 23$, $M = 1$

$$P: \{1,5\}$$

21) $n = 48$, $m = 29$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{19,46\},\{39,43\},\{13,32\},\{14,21\},\{2,45\}, \\ & \{15,29\},\{37,40\},\{11,28\},\{5,35\} \\ R: & \{4,6\},\{12,20\},\{18,34\},\{26,36\},\{30,42\} \end{array}$$

22) $n = 49$, $m = 10$, $s = 11$, $M = 1$

$$\begin{array}{ll} P: & \{1,3\} \\ R: & \{5,29\},\{27,46\},\{17,18\},\{42,45\},\{7,34\}, \\ & \{21,33\},\{35,41\},\{23,37\},\{25,36\},\{15,43\}, \\ & \{19,26\},\{9,14\},\{28,38\} \end{array}$$

23) $n = 50$, $m = 7$, $s = 2$, $M = 1$

$$\begin{array}{ll} P: & \{9,27\},\{6,36\},\{4,49\},\{12,46\},\{23,31\}, \\ & \{38,45\},\{18,20\},\{1,14\},\{19,41\},\{5,44\}, \\ & \{3,32\},\{30,47\} \end{array}$$

24) $n = 51$, $m = 2$, $s = 3$, $M = 1$

$$\begin{array}{ll} P: & \{7,23\},\{1,31\},\{27,47\},\{25,32\},\{9,10\}, \\ & \{42,48\} \\ R: & \{5,30\},\{8,44\},\{12,17\},\{16,24\},\{19,29\}, \\ & \{21,38\},\{34,37\} \end{array}$$

25) $n = 53$, $m = 10$, $s = 13$, $M = 1$

$$P: \quad \{1,3\},\{4,14\}$$

26) $n = 55$, $m = 2$, $s = 7$, $M = 1$

$$\begin{array}{ll} P: & \{1,3\},\{5,18\} \\ R: & \{15,42\},\{49,54\},\{23,44\},\{28,38\},\{11,51\}, \\ & \{31,53\},\{30,47\},\{14,33\},\{29,43\},\{7,37\}, \\ & \{19,39\},\{21,22\},\{35,46\} \end{array}$$

27) $n = 56$, $m = 3$, $s = 3$, $M = 1$

$$\begin{array}{ll} P: & \{30,38\},\{11,44\},\{12,17\},\{39,40\},\{16,18\}, \\ & \{1,53\} \\ R: & \{7,21\},\{10,31\},\{13,35\},\{14,45\},\{19,26\}, \\ & \{22,49\},\{23,42\},\{25,55\},\{27,37\} \end{array}$$

28) $n = 57$, $m = 4$, $s = 4$, $M = 1$

$$\begin{array}{ll} P: & \{13,30\},\{8,31\},\{11,53\},\{35,43\},\{3,7\} \\ R: & \{2,22\},\{9,27\},\{15,45\},\{18,51\},\{19,25\}, \\ & \{23,42\},\{33,54\},\{36,38\} \end{array}$$

29) $n = 58$, $m = 3$, $s = 7$, $M = 1$

$$P: \quad \{1,12\},\{17,30\},\{28,46\},\{41,57\}$$

30) $n = 59$, $m = 3$, $s = 29$, $M = 1$

$$P: \quad \{1,6\}$$

31) $n = 61$, $m = 12$, $s = 15$, $M = 1$

$$P: \quad \{1,3\},\{2,6\}$$

32) $n = 63$, $m = 11$, $s = 5$, $M = 1$

$$\begin{array}{ll} P: & \{7,24\},\{15,38\},\{8,20\},\{32,61\} \\ R: & \{2,9\},\{16,42\},\{17,30\},\{18,43\},\{19,54\}, \\ & \{21,45\},\{22,58\},\{27,48\},\{35,53\},\{36,50\}, \\ & \{46,55\} \end{array}$$

33) $n = 64$, $m = 11$, $s = 4$, $M = 1$

$$\begin{array}{ll} P: & \{21,62\},\{3,17\},\{37,57\},\{5,44\},\{2,20\} \\ R: & \{1,10\},\{6,7\},\{8,15\},\{11,24\},\{13,48\}, \\ & \{16,53\},\{18,58\},\{19,34\},\{30,41\},\{40,56\}, \\ & \{46,54\} \end{array}$$

34) $n = 65$, $m = 7$, $s = 4$, $M = 1$

$$\begin{array}{ll} P: & \{30,33\},\{26,49\},\{48,58\},\{54,63\},\{43,50\} \\ R: & \{1,7\},\{2,10\},\{3,22\},\{5,6\},\{8,28\}, \\ & \{14,42\},\{17,44\},\{21,34\},\{23,35\},\{24,60\}, \\ & \{31,56\},\{38,64\} \end{array}$$

35) $n = 66$, $m = 13$, $s = 5$, $M = 1$

$$\begin{array}{ll} P: & \{3,23\},\{37,56\},\{7,58\},\{24,65\} \\ R: & \{6,18\},\{9,27\},\{10,12\},\{11,51\},\{14,20\}, \\ & \{16,40\},\{21,55\},\{22,50\},\{32,62\},\{36,44\}, \\ & \{42,64\},\{52,63\} \end{array}$$

36) $n = 67$, $m = 4$, $s = 33$, $M = 1$

$$P: \quad \{1,3\}$$

37) $n = 69$, $m = 2$, $s = 5$, $M = 1$

$$P: \quad \{3,44\},\{10,54\},\{56,57\},\{30,67\},\{2,50\}$$
$$R: \quad \{1,23\},\{5,28\},\{13,64\},\{25,49\},\{26,29\},\{27,63\},\{35,46\},\{47,59\},\{52,58\}$$

38) $n = 71$, $m = 2$, $s = 35$, $M = 1$

$$P: \quad \{1,7\}$$

39) $n = 73$, $m = 2$, $s = 9$, $M = 1$

$$P: \quad \{1,3\},\{5,15\},\{9,33\},\{13,35\}$$

40) $n = 74$, $m = 3$, $s = 9$, $M = 1$

$$P: \quad \{1,4\},\{13,56\},\{18,70\},\{61,73\}$$

41) $n = 75$, $m = 17$, $s = 7$, $M = 1$

$$P: \quad \{1,3\},\{7,16\},\{24,43\}$$
$$R: \quad \{25,65\},\{31,35\},\{14,45\},\{60,70\},\{23,71\},\{37,66\},\{9,69\},\{27,72\},\{13,20\},\{2,15\},\{11,29\},\{34,40\},\{10,53\},\{48,68\},\{50,55\},\{5,30\}$$

42) $n = 77$, $m = 2$, $s = 5$, $M = 1$

$$P: \quad \{9,32\},\{1,57\},\{28,47\},\{45,48\},\{40,49\}$$
$$R: \quad \{5,31\},\{10,23\},\{11,43\},\{15,55\},\{20,54\},\{22,30\},\{29,62\},\{33,53\},\{39,61\},\{41,66\},\{44,60\},\{46,73\},\{58,69\}$$

43) $n = 79$, $m = 2$, $s = 39$, $M = 1$

$$P: \quad \{1,3\}$$

44) $n = 83$, $m = 3$, $s = 41$, $M = 1$

$$P: \quad \{1,5\}$$

45) $n = 85$, $m = 22$, $s = 13$, $M = 1$

$$P: \quad \{1,3\},\{2,6\}$$
$$R: \quad \{34,60\},\{39,74\},\{37,68\},\{49,62\},\{45,50\},\{52,75\},\{15,26\},\{10,70\},\{17,80\},\{4,55\},\{30,40\},\{8,25\},\{13,58\},\{31,51\},\{5,20\},\{35,65\}$$

46) $n = 87$, $m = 2$, $s = 11$, $M = 1$

$$P: \quad \{57,74\},\{44,71\},\{14,69\}$$
$$R: \quad \{7,22\},\{9,67\},\{11,24\},\{18,48\},\{29,79\},\{31,47\},\{36,62\},\{37,72\},\{49,85\},\{58,83\}$$

47) $n = 88$, $m = 7$, $s = 5$, $M = 1$

$$P: \quad \{9,67\},\{48,65\},\{23,30\},\{21,69\},\{79,84\},\{38,58\}$$
$$R: \quad \{4,18\},\{5,16\},\{11,24\},\{12,22\},\{19,52\},\{20,86\},\{28,55\},\{32,50\},\{33,35\},\{40,78\},\{45,66\},\{51,80\},\{74,77\}$$

48) $n = 89$, $m = 2$, $s = 11$, $M = 1$

$$P: \quad \{1,3\},\{5,15\},\{9,22\},\{19,65\}$$

49) $n = 93$, $m = 14$, $s = 12$, $M = 1$

$$P: \quad \{35,80\},\{48,85\},\{23,66\}$$
$$R: \quad \{3,77\},\{5,8\},\{18,26\},\{19,49\},\{30,81\},\{31,86\},\{39,70\},\{42,55\},\{46,50\},\{62,88\}$$

50) $n = 95$, $m = 21$, $s = 7$, $M = 1$

$$P: \quad \{20,51\},\{39,63\},\{2,37\},\{81,94\},\{83,90\}$$
$$R: \quad \{3,9\},\{5,24\},\{6,38\},\{8,73\},\{10,67\},\{11,14\},\{13,29\},\{19,41\},\{25,76\},\{31,57\},\{47,62\},\{50,77\}$$

51) $n = 99$, $m = 20$, $s = 8$, $M = 1$

$$P: \quad \{1,3\},\{2,17\},\{21,38\},\{34,41\}$$
$$R: \quad \{72,88\},\{6,30\},\{5,27\},\{7,83\},\{26,81\},\{35,44\},\{77,95\},\{22,33\},\{11,75\},\{15,36\},\{54,58\},\{9,63\},\{18,45\},\{71,90\},\{19,55\},\{25,91\},\{66,76\}$$

52) $n = 103$, $m = 2$, $s = 51$, $M = 1$

$$P: \quad \{1,3\}$$

53) $n = 104$, $m = 11$, $s = 6$, $M = 1$

$$P: \quad \{27,59\},\{12,56\},\{45,48\},\{31,50\},\{70,92\},\{2,47\}$$
$$R: \quad \{1,39\},\{6,35\},\{10,23\},\{11,13\},\{17,78\},\{19,93\},\{21,86\},\{26,83\},\{41,51\},\{65,91\},\{66,73\},\{75,102\},\{81,87\},\{82,97\},\{85,103\}$$

54) $n = 106$, $m = 7$, $s = 13$, $M = 1$

$$P: \quad \{1,3\},\{8,24\},\{82,103\},\{98,105\}$$

55) $n = 107$, $m = 3$, $s = 53$, $M = 1$

$$P: \quad \{1,5\}$$

56) $n = 109$, $m = 2$, $s = 18$, $M = 1$

$$P: \quad \{1,3\},\{50,106\},\{59,108\}$$

57) $n = 111$, $m = 7$, $s = 8$, $M = 1$

$$P: \quad \{2,95\},\{13,80\},\{4,86\},\{36,106\},\{15,70\},\{12,45\}$$
$$R: \quad \{10,18\},\{21,31\},\{32,77\},\{37,64\},\{44,81\},\{54,74\},\{59,97\}$$

58) $n = 123$, $m = 5$, $s = 9$, $M = 1$

$$P: \quad \{76,113\},\{56,116\},\{26,114\},\{45,117\},\{2,6\},\{16,92\}$$
$$R: \quad \{5,121\},\{9,41\},\{25,75\},\{43,82\},\{48,89\},\{51,85\},\{67,79\}$$

59) $n = 125$, $m = 2$, $s = 10$, $M = 1$

$$P: \quad \{1,11\},\{10,67\},\{28,31\},\{29,111\},\{108,115\}$$
$$R: \quad \{14,48\},\{17,78\},\{24,100\},\{25,96\},\{34,59\},\{39,71\},\{47,119\},\{50,77\},\{63,101\},\{75,92\},\{68,118\},\{94,113\}$$

60) $n = 127$, $m = 9$, $s = 63$, $M = 1$

$$P: \quad \{1,3\}$$

61) $n = 131$, $m = 3$, $s = 65$, $M = 1$

$$P: \quad \{1, 6\}$$

62) $n = 133$, $m = 11$, $s = 3$, $M = 1$

$$\begin{array}{ll} P: & \{1,3\},\{2,5\},\{89,124\},\{6,10\},\{7,8\}, \\ & \{80,87\},\{24,69\},\{15,31\},\{56,95\},\{17,30\}, \\ & \{23,105\},\{28,45\},\{29,47\},\{46,98\},\{35,67\}, \\ & \{58,128\},\{40,59\},\{25,85\},\{16,36\},\{12,79\}, \\ & \{19,50\},\{68,93\} \end{array}$$

63) $n = 138$, $m = 19$, $s = 10$, $M = 1$

$$\begin{array}{ll} P: & \{1,3\},\{2,5\},\{4,12\},\{8,81\} \\ R: & \{37,78\},\{7,96\},\{43,115\},\{77,131\},\{41,66\}, \\ & \{47,124\},\{6,89\},\{13,59\},\{34,87\},\{73,92\}, \\ & \{18,48\},\{17,52\},\{23,127\},\{31,91\},\{44,135\}, \\ & \{70,121\},\{83,84\},\{10,104\},\{30,72\},\{65,88\}, \\ & \{67,126\},\{35,53\},\{16,118\},\{22,28\},\{46,94\}, \\ & \{113,130\},\{102,114\},\{109,133\} \end{array}$$

64) $n = 139$, $m = 4$, $s = 69$, $M = 1$

$$P: \quad \{1, 3\}.$$

∎

Proposition 6.8: $A_4(8, 5, [2, 1, 1]) \geq 18$.

Proof: For $n = 8$, the 18 required codewords are:

$$\begin{array}{llll} \langle 3,7,5,2\rangle & \langle 2,5,1,7\rangle & \langle 1,3,6,7\rangle & \langle 2,3,4,1\rangle \\ \langle 0,5,4,2\rangle & \langle 6,7,2,1\rangle & \langle 0,7,6,5\rangle & \langle 4,6,3,2\rangle \\ \langle 0,6,1,4\rangle & \langle 5,6,7,3\rangle & \langle 3,4,0,5\rangle & \langle 0,3,2,6\rangle \\ \langle 4,5,6,1\rangle & \langle 4,7,1,3\rangle & \langle 2,7,0,4\rangle & \langle 0,1,5,3\rangle \\ \langle 1,4,2,0\rangle & \langle 2,6,5,0\rangle. & & \end{array}$$

∎

Proposition 6.9: $A_4(5, 5, [2, 1, 1]) = 2$, $A_4(9, 5, [2, 1, 1]) \geq 27$ and $A_4(13, 5, [2, 1, 1]) \geq 72$.

Proof: For $n = 5$, the 2 required codewords are:

$$\langle 0,1,2,3\rangle \quad \langle 2,3,0,4\rangle$$

For $n = 9$, the 27 required codewords are:

$$\begin{array}{llll} \langle 0,5,4,1\rangle & \langle 1,8,0,3\rangle & \langle 0,6,2,3\rangle & \langle 2,3,0,4\rangle \\ \langle 1,2,6,0\rangle & \langle 0,8,6,4\rangle & \langle 4,5,7,8\rangle & \langle 6,8,1,2\rangle \\ \langle 5,8,2,0\rangle & \langle 1,3,2,7\rangle & \langle 0,2,7,5\rangle & \langle 1,5,8,6\rangle \\ \langle 2,7,3,8\rangle & \langle 0,3,5,8\rangle & \langle 7,8,5,1\rangle & \langle 3,4,8,1\rangle \\ \langle 4,7,1,6\rangle & \langle 2,5,1,3\rangle & \langle 4,6,0,5\rangle & \langle 6,7,8,0\rangle \\ \langle 5,7,6,2\rangle & \langle 0,4,3,7\rangle & \langle 1,4,5,2\rangle & \langle 1,6,3,4\rangle \\ \langle 2,6,4,7\rangle & \langle 3,7,4,5\rangle & \langle 3,8,7,6\rangle & \end{array}$$

For $n = 13$, the 72 required codewords are all quasi-cyclic shifts with length 2 of the following vectors, where the element a keeps fixed under the action of the automorphism group.

$$\begin{array}{llll} \langle a,0,3,2\rangle & \langle a,9,2,7\rangle & \langle 0,11,a,8\rangle & \langle 0,7,2,a\rangle \\ \langle 0,5,4,7\rangle & \langle 0,3,5,11\rangle & \langle 0,4,11,1\rangle & \langle 0,9,6,3\rangle \\ \langle 0,10,8,4\rangle & \langle 1,7,5,2\rangle & \langle 1,3,4,6\rangle & \langle 0,1,9,5\rangle. \end{array}$$

∎

Proposition 6.10: $A_4(6, 5, [2, 1, 1]) = 6$, $A_4(10, 5, [2, 1, 1]) \geq 36$.

Proof: For $n = 6$, the 6 required codewords are:

$$\begin{array}{llll} \langle 0,1,2,3\rangle & \langle 0,2,4,5\rangle & \langle 1,3,5,4\rangle & \langle 2,4,3,1\rangle \\ \langle 3,5,0,2\rangle & \langle 4,5,1,0\rangle & & \end{array}$$

For $n = 10$, the 36 required codewords are:

$$\begin{array}{llll} \langle 8,9,4,1\rangle & \langle 4,8,7,6\rangle & \langle 4,5,2,7\rangle & \langle 1,7,8,4\rangle \\ \langle 5,9,3,4\rangle & \langle 0,8,1,7\rangle & \langle 1,8,6,3\rangle & \langle 1,6,7,0\rangle \\ \langle 1,9,2,6\rangle & \langle 3,7,0,1\rangle & \langle 4,9,6,0\rangle & \langle 6,9,8,7\rangle \\ \langle 6,7,2,3\rangle & \langle 3,9,7,2\rangle & \langle 0,1,5,9\rangle & \langle 2,8,5,4\rangle \\ \langle 1,2,3,7\rangle & \langle 2,7,4,0\rangle & \langle 0,4,8,5\rangle & \langle 4,6,5,1\rangle \\ \langle 0,3,2,4\rangle & \langle 3,4,9,8\rangle & \langle 1,4,0,2\rangle & \langle 0,2,6,1\rangle \\ \langle 7,8,3,9\rangle & \langle 0,6,3,8\rangle & \langle 0,5,7,3\rangle & \langle 5,7,6,8\rangle \\ \langle 3,6,1,9\rangle & \langle 2,3,8,6\rangle & \langle 7,9,1,5\rangle & \langle 5,8,9,0\rangle \\ \langle 0,7,9,6\rangle & \langle 1,3,4,5\rangle & \langle 2,6,9,5\rangle & \langle 2,9,0,8\rangle. \end{array}$$

∎

Proposition 6.11: $A_4(7, 5, [2, 1, 1]) = 10$, $A_4(11, 5, [2, 1, 1]) \geq 48$.

Proof: For $n = 7$, the 10 required codewords are:

$$\begin{array}{llll} \langle 0,1,2,3\rangle & \langle 0,2,4,5\rangle & \langle 0,3,5,6\rangle & \langle 0,4,6,1\rangle \\ \langle 1,4,3,5\rangle & \langle 1,6,4,2\rangle & \langle 2,3,1,4\rangle & \langle 2,6,3,0\rangle \\ \langle 4,5,2,6\rangle & \langle 5,6,1,3\rangle & & \end{array}$$

For $n = 11$, the 48 required codewords are:

$$\begin{array}{llll} \langle 4,6,10,9\rangle & \langle 3,10,2,0\rangle & \langle 0,8,2,9\rangle & \langle 3,8,10,5\rangle \\ \langle 1,8,5,2\rangle & \langle 0,2,1,5\rangle & \langle 6,8,9,1\rangle & \langle 3,4,1,2\rangle \\ \langle 2,8,7,4\rangle & \langle 2,10,8,1\rangle & \langle 6,10,1,7\rangle & \langle 5,8,6,7\rangle \\ \langle 0,10,9,4\rangle & \langle 2,4,0,10\rangle & \langle 7,8,1,3\rangle & \langle 3,9,5,4\rangle \\ \langle 8,9,4,10\rangle & \langle 5,9,1,8\rangle & \langle 4,7,6,1\rangle & \langle 1,4,9,5\rangle \\ \langle 5,7,0,4\rangle & \langle 1,2,3,7\rangle & \langle 8,10,0,6\rangle & \langle 6,9,0,2\rangle \\ \langle 7,10,5,8\rangle & \langle 0,4,5,6\rangle & \langle 4,10,7,3\rangle & \langle 4,8,3,0\rangle \\ \langle 1,3,8,9\rangle & \langle 3,5,7,1\rangle & \langle 2,9,10,3\rangle & \langle 0,6,7,8\rangle \\ \langle 5,10,4,2\rangle & \langle 1,9,7,6\rangle & \langle 0,1,6,10\rangle & \langle 0,3,4,7\rangle \\ \langle 9,10,6,5\rangle & \langle 3,7,9,10\rangle & \langle 0,5,8,3\rangle & \langle 1,6,4,3\rangle \\ \langle 2,5,9,6\rangle & \langle 0,7,10,2\rangle & \langle 4,9,2,7\rangle & \langle 7,9,8,0\rangle \\ \langle 5,6,3,10\rangle & \langle 0,9,3,1\rangle & \langle 2,3,6,8\rangle & \langle 6,7,2,5\rangle. \end{array}$$

∎

VII. Base codewords for CCCs and GDCs with distance 6 and type $[2, 1, 1]$

Proposition 7.1: There exists a $[2, 1, 1]$-GDC(6) of type 2^{3t+1} with size $6t^2 + 2t$ for each $t \in \{3, 4, 5, 6, 7, 8, 9, 11\}$.

Proof: For each $t \in \{3, 4, 5, 6, 7, 8, 9, 11\}$, let $X_t = \mathbb{Z}_{2(3t+1)}$, $\mathcal{G}_t = \{\{i, 3t + 1 + i\} : i \in \mathbb{Z}_{3t+1}\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2, 1, 1]$-GDC(6) of type 2^{3t+1} with size $6t^2 + 2t$, where

1) $t = 3$, $n = 20$, $m = 1$, $s = 1$, $M = 1$

$$P: \langle 0,1,4,7\rangle \quad \langle 0,2,14,15\rangle \quad \langle 0,9,5,17\rangle$$

2) $t = 4$, $n = 26$, $m = 3$, $s = 3$, $M = 1$

$$\begin{array}{l} P: \langle 0,1,5,22\rangle \\ R: \langle 0,2,8,20\rangle \end{array}$$

$\langle 6,10,8,16\rangle$	$\langle 9,15,12,16\rangle$	$\langle 2,12,1,9\rangle$	$\langle 6,14,9,13\rangle$	$\langle 11,13,10,6\rangle$	$\langle 0,4,7,14\rangle$	$\langle 2,15,14,6\rangle$	$\langle 0,15,11,3\rangle$
$\langle 1,6,12,11\rangle$	$\langle 5,12,0,4\rangle$	$\langle 3,11,8,9\rangle$	$\langle 5,10,9,11\rangle$	$\langle 0,1,16,2\rangle$	$\langle 1,3,15,5\rangle$	$\langle 7,8,10,2\rangle$	$\langle 3,13,12,0\rangle$
$\langle 6,7,15,0\rangle$	$\langle 10,15,1,4\rangle$	$\langle 11,12,16,14\rangle$	$\langle 3,16,7,6\rangle$	$\langle 8,14,4,11\rangle$	$\langle 5,13,1,14\rangle$	$\langle 10,12,7,13\rangle$	$\langle 7,11,5,1\rangle$
$\langle 0,8,13,5\rangle$	$\langle 1,4,8,13\rangle$	$\langle 0,9,6,10\rangle$	$\langle 4,16,10,12\rangle$	$\langle 4,9,5,3\rangle$	$\langle 5,15,8,7\rangle$	$\langle 8,16,9,1\rangle$	$\langle 5,6,3,2\rangle$
$\langle 4,11,15,2\rangle$	$\langle 1,9,14,7\rangle$	$\langle 2,16,11,13\rangle$	$\langle 7,13,16,4\rangle$	$\langle 7,14,3,12\rangle$	$\langle 8,12,6,3\rangle$	$\langle 14,16,5,15\rangle$	$\langle 9,13,2,8\rangle$
$\langle 10,14,2,0\rangle$	$\langle 2,3,4,10\rangle$						

TABLE I
CODEWORDS OF AN OPTIMAL $(17,6,[2,1,1])_4$-CODE WITH SIZE 42.

$\langle 3,10,2,15\rangle$	$\langle 14,22,3,4\rangle$	$\langle 7,8,21,9\rangle$	$\langle 20,21,5,7\rangle$	$\langle 5,14,15,1\rangle$	$\langle 15,18,21,10\rangle$	$\langle 5,13,18,20\rangle$	$\langle 0,9,14,15\rangle$
$\langle 8,13,1,15\rangle$	$\langle 6,21,10,14\rangle$	$\langle 4,9,18,8\rangle$	$\langle 17,21,19,8\rangle$	$\langle 8,11,20,6\rangle$	$\langle 11,22,21,0\rangle$	$\langle 2,16,9,3\rangle$	$\langle 5,12,0,4\rangle$
$\langle 3,18,20,22\rangle$	$\langle 3,8,12,14\rangle$	$\langle 14,16,0,11\rangle$	$\langle 15,19,13,7\rangle$	$\langle 20,22,8,13\rangle$	$\langle 5,7,17,2\rangle$	$\langle 0,20,3,12\rangle$	$\langle 4,21,22,15\rangle$
$\langle 12,22,16,7\rangle$	$\langle 14,19,6,18\rangle$	$\langle 5,19,11,21\rangle$	$\langle 10,20,9,11\rangle$	$\langle 7,16,18,15\rangle$	$\langle 1,21,18,2\rangle$	$\langle 2,12,6,1\rangle$	$\langle 6,20,15,19\rangle$
$\langle 12,14,21,20\rangle$	$\langle 1,11,16,9\rangle$	$\langle 9,19,10,2\rangle$	$\langle 4,20,14,2\rangle$	$\langle 19,22,1,17\rangle$	$\langle 10,12,17,19\rangle$	$\langle 8,18,2,16\rangle$	$\langle 0,10,7,21\rangle$
$\langle 0,6,11,2\rangle$	$\langle 2,14,8,7\rangle$	$\langle 6,9,13,22\rangle$	$\langle 3,19,4,0\rangle$	$\langle 1,7,22,20\rangle$	$\langle 3,9,7,1\rangle$	$\langle 0,4,16,13\rangle$	$\langle 4,7,6,10\rangle$
$\langle 12,18,9,13\rangle$	$\langle 2,13,19,4\rangle$	$\langle 1,15,3,6\rangle$	$\langle 8,10,4,5\rangle$	$\langle 11,13,10,12\rangle$	$\langle 0,8,17,22\rangle$	$\langle 17,18,7,11\rangle$	$\langle 3,21,13,11\rangle$
$\langle 2,11,5,18\rangle$	$\langle 4,11,17,3\rangle$	$\langle 2,17,22,10\rangle$	$\langle 1,4,5,19\rangle$	$\langle 15,22,5,9\rangle$	$\langle 5,6,8,3\rangle$	$\langle 16,19,8,12\rangle$	$\langle 15,17,4,14\rangle$
$\langle 1,10,14,13\rangle$	$\langle 16,20,17,1\rangle$	$\langle 9,17,20,5\rangle$	$\langle 12,15,11,8\rangle$	$\langle 10,22,18,6\rangle$	$\langle 7,11,14,19\rangle$	$\langle 5,16,10,22\rangle$	$\langle 1,17,12,0\rangle$
$\langle 7,13,0,3\rangle$	$\langle 13,16,21,6\rangle$	$\langle 0,18,19,5\rangle$	$\langle 3,17,6,16\rangle$	$\langle 13,14,17,9\rangle$	$\langle 2,15,20,0\rangle$	$\langle 9,21,12,16\rangle$	$\langle 6,18,1,4\rangle$

TABLE II
CODEWORDS OF AN OPTIMAL $(23,6,[2,1,1])_4$-CODE WITH SIZE 80.

$\langle 1,3,20,30\rangle$	$\langle 16,34,33,17\rangle$	$\langle 4,31,16,6\rangle$	$\langle 13,19,4,20\rangle$	$\langle 10,21,2,25\rangle$	$\langle 19,25,16,14\rangle$	$\langle 6,18,0,32\rangle$	$\langle 22,24,28,34\rangle$
$\langle 24,33,8,3\rangle$	$\langle 9,10,6,30\rangle$	$\langle 8,10,15,24\rangle$	$\langle 4,23,17,33\rangle$	$\langle 8,31,20,25\rangle$	$\langle 25,27,30,34\rangle$	$\langle 12,17,24,13\rangle$	$\langle 13,29,25,28\rangle$
$\langle 4,10,7,14\rangle$	$\langle 32,33,18,4\rangle$	$\langle 14,27,4,23\rangle$	$\langle 20,33,30,16\rangle$	$\langle 23,29,16,7\rangle$	$\langle 7,16,27,25\rangle$	$\langle 1,31,11,24\rangle$	$\langle 2,22,7,0\rangle$
$\langle 13,23,30,1\rangle$	$\langle 4,12,15,32\rangle$	$\langle 20,27,24,26\rangle$	$\langle 3,9,12,31\rangle$	$\langle 2,24,17,27\rangle$	$\langle 17,32,6,3\rangle$	$\langle 7,30,3,13\rangle$	$\langle 9,29,21,34\rangle$
$\langle 10,20,18,19\rangle$	$\langle 1,22,19,27\rangle$	$\langle 27,29,0,2\rangle$	$\langle 14,22,13,17\rangle$	$\langle 17,21,30,19\rangle$	$\langle 2,9,16,5\rangle$	$\langle 17,29,4,15\rangle$	$\langle 13,18,10,22\rangle$
$\langle 0,31,26,30\rangle$	$\langle 3,25,7,32\rangle$	$\langle 2,20,14,1\rangle$	$\langle 5,8,26,29\rangle$	$\langle 25,33,10,0\rangle$	$\langle 3,8,23,16\rangle$	$\langle 11,16,29,20\rangle$	$\langle 16,32,12,10\rangle$
$\langle 17,33,27,22\rangle$	$\langle 20,25,13,6\rangle$	$\langle 4,11,2,9\rangle$	$\langle 5,23,32,9\rangle$	$\langle 16,18,19,24\rangle$	$\langle 15,24,31,29\rangle$	$\langle 13,31,17,21\rangle$	$\langle 5,30,4,11\rangle$
$\langle 21,22,26,15\rangle$	$\langle 19,28,17,32\rangle$	$\langle 31,33,19,23\rangle$	$\langle 0,32,19,21\rangle$	$\langle 20,21,7,29\rangle$	$\langle 11,17,31,0\rangle$	$\langle 9,25,17,8\rangle$	$\langle 27,32,22,5\rangle$
$\langle 9,24,1,20\rangle$	$\langle 6,11,22,10\rangle$	$\langle 18,30,15,21\rangle$	$\langle 3,6,24,5\rangle$	$\langle 8,22,33,30\rangle$	$\langle 11,19,21,5\rangle$	$\langle 4,21,1,0\rangle$	$\langle 16,22,9,4\rangle$
$\langle 1,8,6,14\rangle$	$\langle 6,28,14,9\rangle$	$\langle 21,34,18,6\rangle$	$\langle 26,30,9,17\rangle$	$\langle 6,29,19,26\rangle$	$\langle 20,23,11,34\rangle$	$\langle 2,25,29,18\rangle$	$\langle 4,24,30,18\rangle$
$\langle 7,32,2,20\rangle$	$\langle 13,27,9,12\rangle$	$\langle 5,7,6,28\rangle$	$\langle 12,22,5,18\rangle$	$\langle 4,20,5,8\rangle$	$\langle 21,27,28,16\rangle$	$\langle 0,5,33,20\rangle$	$\langle 14,29,20,10\rangle$
$\langle 15,20,9,28\rangle$	$\langle 19,30,1,2\rangle$	$\langle 2,3,21,33\rangle$	$\langle 2,31,28,15\rangle$	$\langle 5,31,27,14\rangle$	$\langle 3,27,10,15\rangle$	$\langle 14,16,5,30\rangle$	$\langle 7,19,22,33\rangle$
$\langle 28,33,12,21\rangle$	$\langle 32,34,23,15\rangle$	$\langle 8,21,9,32\rangle$	$\langle 2,12,26,19\rangle$	$\langle 4,34,22,25\rangle$	$\langle 7,14,1,26\rangle$	$\langle 29,31,22,12\rangle$	$\langle 26,34,7,27\rangle$
$\langle 5,24,10,12\rangle$	$\langle 1,26,21,18\rangle$	$\langle 29,32,1,8\rangle$	$\langle 15,23,3,6\rangle$	$\langle 11,34,24,8\rangle$	$\langle 19,34,9,3\rangle$	$\langle 12,30,16,8\rangle$	$\langle 16,28,13,2\rangle$
$\langle 10,31,32,34\rangle$	$\langle 5,15,1,22\rangle$	$\langle 5,18,34,2\rangle$	$\langle 9,14,15,19\rangle$	$\langle 8,27,11,19\rangle$	$\langle 8,17,2,28\rangle$	$\langle 6,12,31,7\rangle$	$\langle 6,30,25,23\rangle$
$\langle 6,13,33,15\rangle$	$\langle 28,30,0,22\rangle$	$\langle 7,15,17,10\rangle$	$\langle 5,21,13,3\rangle$	$\langle 19,23,27,18\rangle$	$\langle 29,30,24,33\rangle$	$\langle 17,18,20,7\rangle$	$\langle 14,18,3,8\rangle$
$\langle 18,28,23,29\rangle$	$\langle 20,22,31,32\rangle$	$\langle 0,7,29,24\rangle$	$\langle 0,10,11,28\rangle$	$\langle 30,34,10,20\rangle$	$\langle 4,26,29,13\rangle$	$\langle 26,28,8,20\rangle$	$\langle 12,34,29,1\rangle$
$\langle 15,25,21,4\rangle$	$\langle 1,10,5,13\rangle$	$\langle 9,18,4,26\rangle$	$\langle 18,27,31,33\rangle$	$\langle 13,24,7,16\rangle$	$\langle 15,33,34,14\rangle$	$\langle 3,29,18,11\rangle$	$\langle 1,17,34,9\rangle$
$\langle 7,11,12,23\rangle$	$\langle 23,26,14,22\rangle$	$\langle 11,28,3,27\rangle$	$\langle 12,21,14,11\rangle$	$\langle 26,33,5,6\rangle$	$\langle 2,23,10,8\rangle$	$\langle 1,33,2,29\rangle$	$\langle 0,14,2,6\rangle$
$\langle 21,23,24,31\rangle$	$\langle 11,18,1,25\rangle$	$\langle 10,17,23,26\rangle$	$\langle 0,1,25,12\rangle$	$\langle 14,34,31,28\rangle$	$\langle 0,8,18,13\rangle$	$\langle 3,13,26,14\rangle$	$\langle 25,26,12,24\rangle$
$\langle 10,12,33,27\rangle$	$\langle 19,26,10,31\rangle$	$\langle 25,28,5,31\rangle$	$\langle 2,6,4,34\rangle$	$\langle 6,27,1,17\rangle$	$\langle 30,32,14,31\rangle$	$\langle 0,3,17,34\rangle$	$\langle 7,31,9,18\rangle$
$\langle 5,17,25,16\rangle$	$\langle 10,22,3,29\rangle$	$\langle 0,15,16,27\rangle$	$\langle 1,28,4,7\rangle$	$\langle 1,16,15,23\rangle$	$\langle 12,20,0,3\rangle$	$\langle 9,33,7,11\rangle$	$\langle 15,19,8,12\rangle$
$\langle 3,4,28,19\rangle$	$\langle 24,32,26,11\rangle$	$\langle 19,24,6,0\rangle$	$\langle 7,8,34,4\rangle$	$\langle 9,32,28,13\rangle$	$\langle 15,26,32,2\rangle$	$\langle 22,25,23,11\rangle$	$\langle 14,24,25,21\rangle$
$\langle 2,13,11,32\rangle$	$\langle 0,9,22,23\rangle$	$\langle 13,34,0,5\rangle$	$\langle 6,16,8,21\rangle$	$\langle 11,15,13,30\rangle$	$\langle 11,14,32,33\rangle$	$\langle 12,23,28,25\rangle$	$\langle 16,26,3,0\rangle$

TABLE III
CODEWORDS OF AN OPTIMAL $(35,6,[2,1,1])_4$-CODE WITH SIZE 192.

3) $t=5$, $n=32$, $m=7$, $s=2$, $M=1$

$$P:\langle 0,1,3,29\rangle \quad \langle 0,6,18,19\rangle$$
$$R:\langle 0,8,17,23\rangle$$

4) $t=6$, $n=38$, $m=7$, $s=3$, $M=1$

$$P:\langle 0,1,3,30\rangle \quad \langle 0,4,9,16\rangle$$

5) $t=7$, $n=44$, $m=3$, $s=2$, $M=1$

$$P:\langle 0,1,5,8\rangle \quad \langle 0,2,39,36\rangle$$
$$R:\langle 0,9,27,40\rangle \quad \langle 0,16,33,26\rangle \quad \langle 0,19,32,30\rangle$$

6) $t=8$, $n=50$, $m=3$, $s=5$, $M=1$

$$P:\langle 0,2,19,33\rangle$$
$$R:\langle 0,14,42,22\rangle \quad \langle 0,20,5,15\rangle \quad \langle 0,24,34,40\rangle$$

7) $t=9$, $n=56$, $m=3$, $s=2$, $M=1$

$$P:\langle 0,1,5,8\rangle \quad \langle 0,2,51,44\rangle \quad \langle 0,25,36,48\rangle$$
$$R:\langle 0,18,45,47\rangle \quad \langle 0,22,9,39\rangle \quad \langle 0,26,10,16\rangle$$

8) $t=11$, $n=68$, $m=11$, $s=3$, $M=1$

$$P:\langle 0,1,43,63\rangle \quad \langle 0,20,46,39\rangle$$
$$R:\langle 0,17,10,9\rangle \quad \langle 0,44,47,25\rangle \quad \langle 0,36,59,6\rangle$$
$$\langle 0,31,29,4\rangle \quad \langle 0,12,45,8\rangle.$$

■

Proposition 7.2: There exists a $[2,1,1]$-GDC(6) of type 3^{2t+1} with size $6t^2+3t$ for each $t \in \{2,3\}$.

Proof: For each $t \in \{2,3\}$, let $X_t = \mathbb{Z}_{3(2t+1)}$, $\mathcal{G}_t = \{\{i, 2t+1+i, 4t+2+i\} : i \in \mathbb{Z}_{2t+1}\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors

respectively. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2,1,1]$-GDC(6) of type 3^{2t+1} with size $6t^2+3t$, where

1) $t=2$, $n=15$, $m=2$, $s=2$, $M=1$

$$P:\langle 0,1,4,12\rangle$$

2) $t=3$, $n=21$, $m=2$, $s=3$, $M=1$

$$P:\langle 0,1,9,13\rangle.$$

■

Proposition 7.3: There exists a $[2,1,1]$-GDC(6) of type 4^t with size $\frac{8t(t-1)}{3}$ for each $t\in\{4,7\}$.

Proof: For each $t\in\{4,7\}$, let $X_t=\mathbb{Z}_{4t}$, $\mathcal{G}_t=\{\{i,t+i,2t+i,3t+i\}:i\in\mathbb{Z}_t\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t,\mathcal{G}_t,\mathcal{C}_t)$ is a $[2,1,1]$-GDC(6) of type 4^t with size $\frac{8t(t-1)}{3}$, where

1) $t=4$, $n=16$, $m=5$, $s=2$, $M=1$

$$P:\langle 0,1,7,10\rangle$$

2) $t=7$, $n=28$, $m=3$, $s=2$, $M=1$

$$P:\langle 0,1,5,17\rangle$$
$$R:\langle 0,2,10,11\rangle\quad\langle 0,6,19,24\rangle.$$

■

Proposition 7.4: There exists a $[2,1,1]$-GDC(6) of type 6^t with size $6t(t-1)$ for each $t\in\{4,5,6,7\}$.

Proof: For each $t\in\{4,5,6,7\}$, let $X_t=\mathbb{Z}_{6t}$, $\mathcal{G}_t=\{\{i,t+i,2t+i,3t+i,4t+i,5t+i\}:i\in\mathbb{Z}_t\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t,\mathcal{G}_t,\mathcal{C}_t)$ is a $[2,1,1]$-GDC(6) of type 6^t with size $6t(t-1)$, where

1) $t=4$, $n=24$, $m=5$, $s=2$, $M=1$

$$P:\langle 0,1,3,22\rangle$$
$$R:\langle 0,7,13,18\rangle$$

2) $t=5$, $n=30$, $m=17$, $s=2$, $M=1$

$$P:\langle 0,3,1,7\rangle\quad\langle 0,6,19,22\rangle$$

3) $t=6$, $n=36$, $m=7$, $s=2$, $M=1$

$$P:\langle 0,1,10,14\rangle$$
$$R:\langle 0,25,4,21\rangle\quad\langle 0,16,33,2\rangle\quad\langle 0,5,28,8\rangle$$

4) $t=7$, $n=42$, $m=11$, $s=3$, $M=1$

$$P:\langle 0,3,4,34\rangle\quad\langle 0,12,29,32\rangle.$$

■

Proposition 7.5: There exists a $[2,1,1]$-GDC(6) of type 7^4 with size 98.

Proof: Let $X=\mathbb{Z}_{28}$, $\mathcal{G}=\{\{i,4+i,8+i,\ldots,24+i\}:i\in\mathbb{Z}_4\}$ and $\mathcal{C}$ be the set of quasi-cyclic shifts of the vectors generated by the following vectors. Then $(X,\mathcal{G},\mathcal{C})$ is a $[2,1,1]$-GDC(6) of type 7^4 with size 98, where $n=28$, $m=5$, $s=2$, $M=2$ and

$$P:\langle 0,13,3,18\rangle\quad\langle 1,12,3,14\rangle$$
$$R:\langle 1,4,18,27\rangle\quad\langle 0,5,26,27\rangle\quad\langle 1,8,2,15\rangle.$$

■

Proposition 7.6: There exists a $[2,1,1]$-GDC(6) of type 10^4 with size 200.

Proof: Let $X=\mathbb{Z}_{40}$, $\mathcal{G}=\{\{i,4+i,8+i,\ldots,36+i\}:i\in\mathbb{Z}_4\}$ and $\mathcal{C}$ be the set of quasi-cyclic shifts of the vectors generated by the following vectors. Then $(X,\mathcal{G},\mathcal{C})$ is a $[2,1,1]$-GDC(6) of type 10^4 with size 200, where $n=40$, $m=3$, $s=3$, $M=2$ and

$$P:\langle 0,13,2,11\rangle\quad\langle 0,3,21,22\rangle$$
$$R:\langle 0,35,1,34\rangle\quad\langle 0,25,10,15\rangle\quad\langle 1,34,11,24\rangle$$
$$\langle 0,5,14,31\rangle.$$

■

Proposition 7.7: There exists a $[2,1,1]$-GDC(6) of type 12^t9^1 with size $12t(2t+1)$ for each $t\in\{4,5,\ldots,15,17,18,19,23\}$.

Proof: For each $t\in\{4,5,\ldots,15,17,18,19,23\}$, let $X_t=\mathbb{Z}_{12t}\cup(\{a,b,c\}\times\mathbb{Z}_3)$, $\mathcal{G}_t=\{\{i,t+i,2t+i,\ldots,11t+i\}:i\in\mathbb{Z}_t\}\cup\{\{a,b,c\}\times\mathbb{Z}_3\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t,\mathcal{G}_t,\mathcal{C}_t)$ is a $[2,1,1]$-GDC(6) of type 12^t9^1 with size $12t(2t+1)$.

1) $t=4$, $n=48$, $m=5$, $s=2$, $M=1$

$$P:\langle 0,3,21,30\rangle$$
$$R:\langle a_0,0,13,35\rangle\quad\langle 5,43,a_0,36\rangle\quad\langle b_0,0,23,25\rangle$$
$$\langle 23,37,b_0,18\rangle\quad\langle c_0,0,7,5\rangle\quad\langle 25,47,c_0,36\rangle$$
$$\langle 0,2,1,19\rangle$$

2) $t=5$, $n=60$, $m=17$, $s=3$, $M=1$

$$P:\langle 0,2,3,6\rangle$$
$$R:\langle a_0,0,52,53\rangle\quad\langle 17,46,a_0,0\rangle\quad\langle b_0,0,46,59\rangle$$
$$\langle 4,41,b_0,0\rangle\quad\langle c_0,0,44,7\rangle\quad\langle 5,52,c_0,33\rangle$$
$$\langle 0,24,12,33\rangle\quad\langle 0,21,32,18\rangle$$

3) $t=6$, $n=72$, $m=19$, $s=2$, $M=1$

$$P:\langle 0,9,50,31\rangle\quad\langle 0,7,33,10\rangle\quad\langle 0,67,34,69\rangle$$
$$R:\langle a_0,0,20,28\rangle\quad\langle 8,61,a_0,45\rangle\quad\langle b_0,0,32,52\rangle$$
$$\langle 55,59,b_0,12\rangle\quad\langle c_0,0,35,40\rangle\quad\langle 29,37,c_0,12\rangle$$
$$\langle 0,1,17,44\rangle$$

4) $t=7$, $n=84$, $m=5$, $s=3$, $M=1$

$$P:\langle 0,2,20,71\rangle\quad\langle 0,8,83,66\rangle$$
$$R:\langle a_0,0,65,73\rangle\quad\langle 16,77,a_0,78\rangle\quad\langle b_0,0,68,25\rangle$$
$$\langle 7,74,b_0,54\rangle\quad\langle c_0,0,13,53\rangle\quad\langle 25,68,c_0,30\rangle$$
$$\langle 0,55,31,81\rangle\quad\langle 0,12,15,48\rangle\quad\langle 0,33,37,57\rangle$$

5) $t=8$, $n=96$, $m=77$, $s=4$, $M=1$

$$P:\langle 0,13,87,46\rangle\quad\langle 0,23,62,81\rangle$$
$$R:\langle a_0,0,1,68\rangle\quad\langle 26,31,a_0,51\rangle\quad\langle b_0,0,95,28\rangle$$
$$\langle 34,41,b_0,78\rangle\quad\langle c_0,0,59,76\rangle\quad\langle 28,47,c_0,3\rangle$$
$$\langle 0,66,78,84\rangle\quad\langle 0,54,60,90\rangle\quad\langle 0,35,4,31\rangle$$

6) $t=9$, $n=108$, $m=7$, $s=3$, $M=1$

$$P:\langle 0,25,39,76\rangle\quad\langle 0,24,55,77\rangle\quad\langle 0,95,105,80\rangle$$
$$\langle 0,30,4,8\rangle$$
$$R:\langle a_0,0,106,59\rangle\quad\langle 82,98,a_0,9\rangle\quad\langle b_0,0,79,74\rangle$$
$$\langle 23,46,b_0,72\rangle\quad\langle c_0,0,50,73\rangle\quad\langle 22,86,c_0,18\rangle$$
$$\langle 0,43,89,29\rangle$$

7) $t = 10,\ n = 120,\ m = 113,\ s = 3,\ M = 1$

$P : \langle 0, 82, 33, 86\rangle$ $\langle 0, 29, 117, 78\rangle$ $\langle 0, 77, 79, 73\rangle$
$R : \langle a_0, 0, 46, 32\rangle$ $\langle 37, 44, a_0, 78\rangle$ $\langle b_0, 0, 109, 74\rangle$
$\langle 34, 86, b_0, 99\rangle$ $\langle c_0, 0, 8, 55\rangle$ $\langle 38, 43, c_0, 54\rangle$
$\langle 0, 15, 99, 51\rangle$ $\langle 0, 114, 63, 12\rangle$ $\langle 0, 96, 42, 87\rangle$
$\langle 0, 81, 25, 56\rangle$ $\langle 0, 75, 27, 3\rangle$ $\langle 0, 35, 23, 22\rangle$

8) $t = 11,\ n = 132,\ m = 5,\ s = 5,\ M = 1$

$P : \langle 0, 62, 90, 104\rangle$ $\langle 0, 16, 129, 81\rangle$ $\langle 0, 120, 19, 27\rangle$
$R : \langle a_0, 0, 125, 67\rangle$ $\langle 62, 112, a_0, 87\rangle$ $\langle b_0, 0, 7, 35\rangle$
$\langle 61, 71, b_0, 120\rangle$ $\langle c_0, 0, 97, 17\rangle$ $\langle 0, 14, 85, 43\rangle$
$\langle 0, 106, 83, 119\rangle$ $\langle 56, 58, c_0, 15\rangle$

9) $t = 12,\ n = 144,\ m = 11,\ s = 3,\ M = 1$

$P : \langle 0, 62, 77, 49\rangle$ $\langle 0, 44, 118, 5\rangle$ $\langle 0, 27, 37, 136\rangle$
$\langle 0, 16, 67, 35\rangle$
$R : \langle a_0, 0, 23, 142\rangle$ $\langle 10, 86, a_0, 132\rangle$ $\langle b_0, 0, 88, 50\rangle$
$\langle 5, 25, b_0, 108\rangle$ $\langle c_0, 0, 53, 91\rangle$ $\langle 16, 47, c_0, 24\rangle$
$\langle 0, 114, 95, 104\rangle$ $\langle 0, 90, 79, 61\rangle$ $\langle 0, 63, 141, 138\rangle$
$\langle 0, 137, 66, 86\rangle$ $\langle 0, 102, 69, 143\rangle$ $\langle 0, 28, 34, 98\rangle$
$\langle 0, 18, 57, 89\rangle$

10) $t = 13,\ n = 156,\ m = 37,\ s = 5,\ M = 1$

$P : \langle 0, 50, 29, 125\rangle$ $\langle 0, 154, 100, 54\rangle$ $\langle 0, 148, 49, 155\rangle$
$R : \langle 0, 48, 69, 25\rangle$ $\langle 0, 90, 145, 17\rangle$ $\langle 0, 14, 73, 45\rangle$
$\langle 55, 116, b_0, 135\rangle$ $\langle c_0, 0, 115, 107\rangle$ $\langle 0, 12, 99, 9\rangle$
$\langle 68, 139, c_0, 144\rangle$ $\langle 0, 24, 96, 84\rangle$ $\langle 0, 120, 105, 43\rangle$
$\langle 136, 140, a_0, 111\rangle$ $\langle a_0, 0, 118, 11\rangle$ $\langle b_0, 0, 38, 109\rangle$

11) $t = 14,\ n = 168,\ m = 103,\ s = 5,\ M = 1$

$P : \langle 0, 25, 64, 19\rangle$ $\langle 0, 46, 29, 90\rangle$ $\langle 0, 38, 61, 125\rangle$
$R : \langle b_0, 0, 76, 2\rangle$ $\langle 0, 60, 48, 72\rangle$ $\langle 52, 59, a_0, 111\rangle$
$\langle 22, 113, b_0, 30\rangle$ $\langle c_0, 0, 41, 4\rangle$ $\langle 35, 154, c_0, 138\rangle$
$\langle 0, 93, 136, 102\rangle$ $\langle 0, 45, 150, 65\rangle$ $\langle 0, 24, 124, 104\rangle$
$\langle 0, 51, 133, 11\rangle$ $\langle 0, 69, 96, 132\rangle$ $\langle 0, 147, 120, 145\rangle$
$\langle a_0, 0, 148, 86\rangle$ $\langle 0, 3, 36, 35\rangle$

12) $t = 15,\ n = 180,\ m = 7,\ s = 4,\ M = 1$

$P : \langle 0, 153, 96, 12\rangle$ $\langle 0, 161, 3, 177\rangle$ $\langle 0, 109, 116, 14\rangle$
$\langle 0, 6, 82, 38\rangle$ $\langle 0, 41, 142, 5\rangle$
$R : \langle a_0, 0, 68, 13\rangle$ $\langle 4, 71, a_0, 162\rangle$ $\langle b_0, 0, 4, 170\rangle$
$\langle 5, 145, b_0, 165\rangle$ $\langle c_0, 0, 79, 119\rangle$ $\langle 4, 134, c_0, 135\rangle$
$\langle 0, 80, 136, 10\rangle$ $\langle 0, 54, 179, 26\rangle$ $\langle 0, 169, 17, 134\rangle$
$\langle 0, 18, 2, 115\rangle$ $\langle 0, 77, 52, 70\rangle$

13) $t = 17,\ n = 204,\ m = 41,\ s = 8,\ M = 1$

$P : \langle 0, 3, 50, 125\rangle$ $\langle 0, 126, 37, 55\rangle$ $\langle 0, 105, 14, 191\rangle$
$R : \langle a_0, 0, 20, 52\rangle$ $\langle 65, 73, a_0, 9\rangle$ $\langle b_0, 0, 164, 4\rangle$
$\langle 41, 169, b_0, 141\rangle$ $\langle c_0, 0, 116, 64\rangle$ $\langle 38, 82, c_0, 66\rangle$
$\langle 0, 108, 56, 104\rangle$ $\langle 0, 80, 112, 172\rangle$ $\langle 0, 168, 184, 192\rangle$
$\langle 0, 60, 72, 180\rangle$ $\langle 0, 48, 132, 88\rangle$

14) $t = 18,\ n = 216,\ m = 23,\ s = 4,\ M = 1$

$P : \langle 0, 89, 25, 38\rangle$ $\langle 0, 118, 61, 78\rangle$ $\langle 0, 172, 185, 84\rangle$
$\langle 0, 163, 88, 140\rangle$ $\langle 0, 29, 105, 203\rangle$ $\langle 0, 86, 48, 109\rangle$
$R : \langle a_0, 0, 65, 142\rangle$ $\langle 5, 151, a_0, 129\rangle$ $\langle 0, 1, 27, 190\rangle$
$\langle 2, 151, b_0, 201\rangle$ $\langle c_0, 0, 22, 41\rangle$ $\langle 7, 86, c_0, 39\rangle$
$\langle b_0, 0, 167, 37\rangle$ $\langle 0, 153, 147, 92\rangle$ $\langle 0, 45, 60, 195\rangle$
$\langle 0, 154, 12, 129\rangle$ $\langle 0, 99, 192, 96\rangle$ $\langle 0, 207, 177, 64\rangle$
$\langle 0, 81, 31, 183\rangle$

15) $t = 19,\ n = 228,\ m = 43,\ s = 7,\ M = 1$

$P : \langle 0, 74, 43, 139\rangle$ $\langle 0, 88, 226, 186\rangle$ $\langle 0, 21, 156, 93\rangle$
$\langle 0, 23, 196, 31\rangle$
$R : \langle 0, 40, 82, 126\rangle$ $\langle 14, 142, a_0, 93\rangle$ $\langle b_0, 0, 67, 149\rangle$
$\langle 29, 112, b_0, 201\rangle$ $\langle c_0, 0, 97, 32\rangle$ $\langle 28, 74, c_0, 189\rangle$
$\langle 0, 104, 210, 50\rangle$ $\langle 0, 63, 51, 12\rangle$ $\langle 0, 27, 168, 87\rangle$
$\langle a_0, 0, 203, 157\rangle$ $\langle 0, 70, 131, 71\rangle$

16) $t = 23,\ n = 276,\ m = 7,\ s = 7,\ M = 1$

$P : \langle 0, 8, 86, 114\rangle$ $\langle 0, 200, 257, 43\rangle$ $\langle 0, 58, 261, 125\rangle$
$\langle 0, 228, 211, 105\rangle$
$R : \langle 0, 51, 88, 15\rangle$ $\langle 0, 2, 170, 165\rangle$ $\langle b_0, 0, 109, 83\rangle$
$\langle 13, 161, b_0, 240\rangle$ $\langle c_0, 0, 251, 181\rangle$ $\langle 11, 187, c_0, 216\rangle$
$\langle 16, 134, a_0, 204\rangle$ $\langle a_0, 0, 89, 13\rangle$ $\langle 0, 80, 87, 145\rangle$
$\langle 0, 55, 126, 210\rangle$ $\langle 0, 98, 179, 30\rangle$ $\langle 0, 249, 75, 162\rangle$
$\langle 0, 224, 49, 167\rangle$ $\langle 0, 72, 108, 90\rangle$ $\langle 0, 214, 147, 1\rangle$
$\langle 0, 252, 110, 142\rangle$ $\langle 0, 14, 201, 68\rangle$ $\langle 0, 64, 233, 149\rangle$
$\langle 0, 104, 195, 215\rangle$.

■

Proposition 7.8: There exists a $[2, 1, 1]$-GDC(6) of type $12^t 15^1$ with size $12t(2t + 3)$ for each $t \in \{7, 8, \dots, 15\}$.

Proof: For each $t \in \{7, 8, \dots, 15\}$, let $X_t = \mathbb{Z}_{12t} \cup (\{a, b, c, d, e\} \times \mathbb{Z}_3)$, $\mathcal{G}_t = \{\{i, t+i, 2t+i, \dots, 11t+i\} : i \in \mathbb{Z}_t\} \cup \{\{a, b, c, d, e\} \times \mathbb{Z}_3\}$ and $\mathcal{C}_t$ be the set of cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2, 1, 1]$-GDC(6) of type $12^t 15^1$ with size $12t(2t + 3)$.

1) $t = 7,\ n = 84,\ m = 11,\ s = 3,\ M = 1$

$P : \langle 0, 39, 36, 3\rangle$ $\langle 0, 18, 52, 83\rangle$
$R : \langle b_0, 0, 25, 50\rangle$ $\langle 17, 22, a_0, 0\rangle$ $\langle a_0, 0, 16, 8\rangle$
$\langle e_0, 0, 55, 23\rangle$ $\langle c_0, 0, 37, 32\rangle$ $\langle 49, 59, c_0, 78\rangle$
$\langle 79, 83, d_0, 12\rangle$ $\langle 4, 68, b_0, 45\rangle$ $\langle 2, 28, e_0, 3\rangle$
$\langle d_0, 0, 11, 22\rangle$ $\langle 0, 40, 71, 2\rangle$

2) $t = 8,\ n = 96,\ m = 5,\ s = 3,\ M = 1$

$P : \langle 0, 6, 59, 93\rangle$ $\langle 0, 1, 11, 38\rangle$
$R : \langle a_0, 0, 20, 19\rangle$ $\langle 20, 88, a_0, 9\rangle$ $\langle b_0, 0, 34, 41\rangle$
$\langle 13, 95, b_0, 78\rangle$ $\langle c_0, 0, 67, 62\rangle$ $\langle 25, 47, c_0, 51\rangle$
$\langle e_0, 0, 70, 29\rangle$ $\langle 0, 60, 3, 69\rangle$ $\langle 17, 64, e_0, 60\rangle$
$\langle 19, 71, d_0, 21\rangle$ $\langle d_0, 0, 23, 76\rangle$ $\langle 0, 84, 15, 33\rangle$
$\langle 0, 18, 31, 75\rangle$

3) $t = 9,\ n = 108,\ m = 5,\ s = 3,\ M = 1$

$P : \langle 0, 58, 39, 42\rangle$ $\langle 0, 83, 41, 105\rangle$ $\langle 0, 79, 94, 57\rangle$
$R : \langle a_0, 0, 26, 43\rangle$ $\langle 58, 62, a_0, 78\rangle$ $\langle b_0, 0, 88, 14\rangle$
$\langle 56, 61, b_0, 96\rangle$ $\langle c_0, 0, 59, 7\rangle$ $\langle 34, 35, c_0, 102\rangle$
$\langle 31, 92, d_0, 3\rangle$ $\langle e_0, 0, 101, 76\rangle$ $\langle 22, 86, e_0, 33\rangle$
$\langle 0, 100, 48, 52\rangle$ $\langle d_0, 0, 70, 95\rangle$ $\langle 0, 24, 12, 73\rangle$

4) $t = 10,\ n = 120,\ m = 43,\ s = 3,\ M = 1$

$P : \langle 0, 68, 61, 39\rangle$ $\langle 0, 104, 58, 102\rangle$ $\langle 0, 107, 14, 23\rangle$
$R : \langle a_0, 0, 55, 11\rangle$ $\langle 67, 110, a_0, 102\rangle$ $\langle b_0, 0, 8, 1\rangle$
$\langle 55, 80, b_0, 6\rangle$ $\langle c_0, 0, 98, 85\rangle$ $\langle 0, 63, 72, 105\rangle$
$\langle 64, 95, d_0, 102\rangle$ $\langle 0, 33, 45, 51\rangle$ $\langle 50, 55, e_0, 36\rangle$
$\langle 58, 62, c_0, 111\rangle$ $\langle d_0, 0, 65, 97\rangle$ $\langle e_0, 0, 67, 119\rangle$
$\langle 0, 114, 48, 15\rangle$ $\langle 0, 24, 99, 93\rangle$

5) $t=11$, $n=132$, $m=5$, $s=4$, $M=1$

$$\begin{array}{lll} P: \langle 0,102,82,61\rangle & \langle 0,7,16,124\rangle & \langle 0,12,3,50\rangle \\ R: \langle d_0,0,74,109\rangle & \langle a_0,0,25,131\rangle & \langle b_0,0,1,128\rangle \\ \langle 34,44,b_0,6\rangle & \langle c_0,0,106,2\rangle & \langle 14,79,c_0,54\rangle \\ \langle 19,98,d_0,3\rangle & \langle 25,128,a_0,96\rangle & \langle 17,34,e_0,102\rangle \\ \langle e_0,0,127,5\rangle & \langle 0,108,27,52\rangle & \langle 0,34,81,129\rangle \\ \langle 0,6,119,19\rangle & & \end{array}$$

6) $t=12$, $n=144$, $m=101$, $s=3$, $M=1$

$$\begin{array}{lll} P: \langle 0,8,17,3\rangle & \langle 0,14,57,100\rangle & \langle 0,16,113,127\rangle \\ R: \langle a_0,0,79,77\rangle & \langle 14,103,a_0,84\rangle & \langle e_0,0,47,52\rangle \\ \langle 20,73,b_0,21\rangle & \langle c_0,0,98,10\rangle & \langle 11,61,c_0,60\rangle \\ \langle 11,70,d_0,132\rangle & \langle 0,42,124,63\rangle & \langle 11,13,e_0,120\rangle \\ \langle 0,93,66,27\rangle & \langle d_0,0,35,22\rangle & \langle 0,76,105,101\rangle \\ \langle 0,30,58,99\rangle & \langle 0,13,7,44\rangle & \langle 0,126,122,116\rangle \\ \langle 0,54,87,119\rangle & \langle b_0,0,106,95\rangle & \langle 0,61,135,67\rangle \end{array}$$

7) $t=13$, $n=156$, $m=7$, $s=6$, $M=1$

$$\begin{array}{lll} P: \langle 0,56,21,135\rangle & \langle 0,12,11,131\rangle & \langle 0,114,61,57\rangle \\ R: \langle a_0,0,94,122\rangle & \langle 47,55,a_0,69\rangle & \langle b_0,0,154,62\rangle \\ \langle 59,91,b_0,69\rangle & \langle d_0,0,110,34\rangle & \langle 49,53,c_0,99\rangle \\ \langle 64,104,d_0,6\rangle & \langle e_0,0,106,86\rangle & \langle 23,151,e_0,141\rangle \\ \langle c_0,0,38,82\rangle & \langle 0,68,142,70\rangle & \end{array}$$

8) $t=14$, $n=168$, $m=101$, $s=5$, $M=1$

$$\begin{array}{lll} P: \langle 0,39,37,34\rangle & \langle 0,19,120,123\rangle & \langle 0,54,13,58\rangle \\ R: \langle a_0,0,77,22\rangle & \langle 20,28,a_0,60\rangle & \langle b_0,0,62,31\rangle \\ \langle 29,94,b_0,45\rangle & \langle d_0,0,7,128\rangle & \langle c_0,0,44,49\rangle \\ \langle 22,98,d_0,153\rangle & \langle 20,55,c_0,162\rangle & \langle 23,139,e_0,159\rangle \\ \langle 0,147,81,111\rangle & \langle 0,60,69,2\rangle & \langle 0,73,1,66\rangle \\ \langle 0,27,63,17\rangle & \langle 0,122,45,11\rangle & \langle e_0,0,38,124\rangle \\ \langle 0,12,117,155\rangle & & \end{array}$$

9) $t=15$, $n=180$, $m=37$, $s=4$, $M=1$

$$\begin{array}{lll} P: \langle 0,52,58,49\rangle & \langle 0,61,37,54\rangle & \langle 0,68,134,31\rangle \\ \langle 0,74,167,118\rangle & \langle 0,81,24,28\rangle & \\ R: \langle a_0,0,53,43\rangle & \langle 41,61,a_0,171\rangle & \langle b_0,0,17,85\rangle \\ \langle 59,64,b_0,24\rangle & \langle c_0,0,113,79\rangle & \langle 85,140,c_0,0\rangle \\ \langle d_0,0,151,122\rangle & \langle 58,173,d_0,93\rangle & \langle e_0,0,149,7\rangle \\ \langle 43,53,e_0,123\rangle & \langle 0,36,77,50\rangle & \langle 0,25,3,111\rangle \\ \langle 0,108,89,147\rangle. & & \end{array}$$

■

Proposition 7.9: There exists a $[2,1,1]$-GDC(6) of type 13^4 with size 338.

Proof: Let $X=\mathbb{Z}_{52}$, $\mathcal{G}=\{\{i,4+i,8+i,\dots,48+i\}: i\in\mathbb{Z}_4\}$ and $\mathcal{C}$ be the set of quasi-cyclic shifts of the vectors generated by the following vectors. Then $(X,\mathcal{G},\mathcal{C})$ is a $[2,1,1]$-GDC(6) of type 13^4 with size 338, where $n=52$, $m=5$, $s=3$, $M=2$, and

$$\begin{array}{lll} P: \langle 0,5,3,46\rangle & \langle 1,20,22,47\rangle & \langle 0,7,18,41\rangle \\ R: \langle 0,1,11,26\rangle & \langle 0,13,39,42\rangle & \langle 0,17,30,47\rangle \\ \langle 0,43,14,29\rangle. & & \end{array}$$

■

Proposition 7.10: There exists a $[2,1,1]$-GDC(6) of type 22^4 with size 968.

Proof: Let $X=\mathbb{Z}_{88}$, $\mathcal{G}=\{\{i,4+i,8+i,\dots,84+i\}: i\in\mathbb{Z}_4\}$ and $\mathcal{C}$ be the set of quasi-cyclic shifts of the vectors generated by the following vectors. Then $(X,\mathcal{G},\mathcal{C})$ is a $[2,1,1]$-GDC(6) of type 22^4 with size 968, where $n=88$, $m=5$, $s=3$, $M=2$, and

$$\begin{array}{lll} P: \langle 0,9,23,26\rangle & \langle 0,1,3,62\rangle & \\ R: \langle 0,77,67,22\rangle & \langle 0,69,55,38\rangle & \langle 0,51,82,21\rangle \\ \langle 0,7,66,13\rangle & \langle 0,35,18,65\rangle. & \end{array}$$

■

Proposition 7.11: There exists a $[2,1,1]$-GDC(6) of type $1^{12}2^1$ with size 28.

Proof: Let $X=\mathbb{Z}_{14}$, and $\mathcal{G}_t=\{\{0,1\}\}\cup\{\{i\}: i\in\mathbb{Z}_{14}\setminus\{0,1\}\}$. Then $(X,\mathcal{G},\mathcal{C})$ is a $[2,1,1]$-GDC(6) of type $1^{12}2^1$ with size 28, where $\mathcal{C}$ is the set of

$$\begin{array}{llll} \langle 4,6,7,11\rangle & \langle 1,5,11,7\rangle & \langle 2,10,8,7\rangle & \langle 5,6,0,8\rangle \\ \langle 8,11,5,10\rangle & \langle 8,12,6,2\rangle & \langle 10,12,9,5\rangle & \langle 0,9,11,12\rangle \\ \langle 3,7,1,10\rangle & \langle 1,13,2,3\rangle & \langle 1,6,10,12\rangle & \langle 2,5,12,13\rangle \\ \langle 3,6,13,9\rangle & \langle 4,10,13,1\rangle & \langle 0,8,7,13\rangle & \langle 11,12,3,1\rangle \\ \langle 2,9,1,6\rangle & \langle 0,4,2,5\rangle & \langle 9,13,10,8\rangle & \langle 1,8,4,9\rangle \\ \langle 7,13,5,6\rangle & \langle 5,9,3,4\rangle & \langle 7,11,2,9\rangle & \langle 0,10,3,6\rangle \\ \langle 3,4,8,12\rangle & \langle 2,3,0,11\rangle & \langle 11,13,4,0\rangle & \langle 7,12,0,4\rangle. \end{array}$$

■

Proposition 7.12: There exists a $[2,1,1]$-GDC(6) of type 1^t11^1 with size $6u^2+20u$, where $6u=t$ for each $t\in\{30,36,54,66,78\}$.

Proof: For each $t\in\{30,36,54,66,78\}$, let $X_t=\mathbb{Z}_t\cup(\{a,b,c,d,e\}\times\mathbb{Z}_2)\cup\{f\}$, $\mathcal{G}_t=\{\{i\}: i\in\mathbb{Z}_t\}\cup\{\{a,b,c,d,e\}\times\mathbb{Z}_2\cup\{f\}\}$ and $\mathcal{C}_t$ be the set of quasi-cyclic shifts of the vectors generated by the following vectors respectively. Then $(X_t,\mathcal{G}_t,\mathcal{C}_t)$ is a $[2,1,1]$-GDC(6) of type 1^t11^1 with size $6u^2+20u$, where $6u=t$ and

1) $t=30$, $n=30$, $m=1$, $s=1$, $M=3$

$$\begin{array}{lll} P: \langle a_0,24,7,2\rangle & \langle 1,21,a_0,0\rangle & \langle a_0,28,29,3\rangle \\ \langle 2,11,a_0,28\rangle & \langle b_0,15,13,10\rangle & \langle 1,20,b_0,15\rangle \\ \langle b_0,0,23,2\rangle & \langle 4,29,b_0,12\rangle & \langle c_0,6,5,27\rangle \\ \langle 3,29,c_0,14\rangle & \langle c_0,26,19,4\rangle & \langle 1,22,c_0,24\rangle \\ \langle d_0,2,29,4\rangle & \langle 1,11,d_0,18\rangle & \langle d_0,12,19,9\rangle \\ \langle 3,22,d_0,8\rangle & \langle e_0,14,17,28\rangle & \langle 3,17,e_0,18\rangle \\ \langle e_0,6,9,7\rangle & \langle 1,4,e_0,8\rangle & \langle f,11,0,10\rangle \\ \langle 0,4,f,17\rangle & \langle 0,24,12,16\rangle & \langle 1,7,19,29\rangle \\ \langle 2,8,0,20\rangle & & \end{array}$$

2) $t=36$, $n=36$, $m=1$, $s=1$, $M=3$

$$\begin{array}{lll} P: \langle a_0,24,13,14\rangle & \langle 1,3,a_0,6\rangle & \langle a_0,28,15,35\rangle \\ \langle 2,35,a_0,4\rangle & \langle b_0,12,19,16\rangle & \langle 1,2,b_0,27\rangle \\ \langle b_0,9,8,11\rangle & \langle 4,23,b_0,18\rangle & \langle c_0,30,5,27\rangle \\ \langle 1,4,c_0,0\rangle & \langle c_0,26,22,1\rangle & \langle 3,35,c_0,26\rangle \\ \langle d_0,0,15,10\rangle & \langle 1,17,d_0,30\rangle & \langle d_0,32,17,25\rangle \\ \langle 3,22,d_0,20\rangle & \langle e_0,23,2,6\rangle & \langle 1,14,e_0,23\rangle \\ \langle e_0,10,21,1\rangle & \langle 0,9,e_0,22\rangle & \langle f,0,31,5\rangle \\ \langle 1,29,f,9\rangle & \langle 0,24,8,6\rangle & \langle 1,31,10,26\rangle \\ \langle 2,9,30,3\rangle & \langle 2,28,32,16\rangle & \langle 0,16,28,1\rangle \\ \langle 2,14,24,20\rangle & & \end{array}$$

3) $t = 54,\ n = 54,\ m = 5,\ s = 6,\ M = 3$

$$\begin{array}{lll} P:\langle 0,1,2,3\rangle & & \\ R:\langle a_0,30,19,4\rangle & \langle 1,23,a_0,38\rangle & \langle a_0,8,39,41\rangle \\ \langle 3,16,a_0,48\rangle & \langle b_0,24,33,47\rangle & \langle 1,52,b_0,12\rangle \\ \langle b_0,38,40,19\rangle & \langle 3,29,b_0,2\rangle & \langle c_0,18,22,25\rangle \\ \langle 1,47,c_0,50\rangle & \langle c_0,2,27,23\rangle & \langle 3,40,c_0,30\rangle \\ \langle d_0,42,29,37\rangle & \langle 1,5,d_0,21\rangle & \langle d_0,45,10,20\rangle \\ \langle 2,16,d_0,0\rangle & \langle e_0,0,52,38\rangle & \langle 1,46,e_0,27\rangle \\ \langle e_0,15,29,31\rangle & \langle 2,47,e_0,6\rangle & \langle f,31,24,5\rangle \\ \langle 0,32,f,22\rangle & \langle 0,42,36,6\rangle & \langle 1,31,20,13\rangle \\ \langle 1,22,40,49\rangle & \langle 1,14,6,43\rangle & \langle 2,26,8,45\rangle \\ \langle 1,17,11,16\rangle & \langle 0,24,35,44\rangle & \langle 1,35,7,53\rangle \\ \langle 2,14,53,25\rangle & & \end{array}$$

4) $t = 66,\ n = 66,\ m = 17,\ s = 4,\ M = 3$

$$\begin{array}{lll} P:\langle 0,1,2,3\rangle & \langle 0,5,6,10\rangle & \\ R:\langle a_0,42,22,13\rangle & \langle 1,11,a_0,44\rangle & \langle a_0,26,17,39\rangle \\ \langle 3,64,a_0,36\rangle & \langle b_0,24,39,2\rangle & \langle 1,64,b_0,47\rangle \\ \langle b_0,59,58,1\rangle & \langle 2,21,b_0,0\rangle & \langle c_0,42,32,17\rangle \\ \langle 1,46,c_0,12\rangle & \langle c_0,51,49,34\rangle & \langle 2,5,c_0,57\rangle \\ \langle d_0,18,26,46\rangle & \langle 1,16,d_0,9\rangle & \langle d_0,63,29,13\rangle \\ \langle 2,53,d_0,18\rangle & \langle e_0,14,23,18\rangle & \langle 1,17,e_0,56\rangle \\ \langle e_0,63,1,4\rangle & \langle 0,22,e_0,57\rangle & \langle f,36,10,32\rangle \\ \langle 1,29,f,57\rangle & \langle 0,54,43,48\rangle & \langle 1,19,31,59\rangle \\ \langle 1,10,24,37\rangle & \langle 1,7,40,27\rangle & \langle 1,35,55,65\rangle \\ \langle 1,53,23,63\rangle & \langle 2,20,42,64\rangle & \langle 2,26,49,28\rangle \\ \langle 1,25,62,32\rangle & \langle 0,23,11,35\rangle & \langle 2,62,37,23\rangle \\ \langle 0,39,63,53\rangle & \langle 0,20,31,65\rangle & \end{array}$$

5) $t = 78,\ n = 78,\ m = 11,\ s = 4,\ M = 3$

$$\begin{array}{lll} P:\langle 0,1,2,9\rangle & \langle 0,44,62,56\rangle & \langle 0,61,66,19\rangle \\ R:\langle a_0,18,41,7\rangle & \langle 1,11,a_0,56\rangle & \langle a_0,44,28,3\rangle \\ \langle 3,10,a_0,6\rangle & \langle b_0,6,32,46\rangle & \langle 1,52,b_0,42\rangle \\ \langle b_0,75,31,29\rangle & \langle 2,23,b_0,21\rangle & \langle c_0,72,67,68\rangle \\ \langle 1,35,c_0,21\rangle & \langle c_0,3,16,41\rangle & \langle 2,16,c_0,72\rangle \\ \langle d_0,72,27,29\rangle & \langle 1,20,d_0,46\rangle & \langle d_0,28,13,20\rangle \\ \langle 3,53,d_0,66\rangle & \langle e_0,48,27,52\rangle & \langle 1,8,e_0,47\rangle \\ \langle e_0,29,2,49\rangle & \langle 3,58,e_0,18\rangle & \langle f,15,46,56\rangle \\ \langle 1,5,f,27\rangle & \langle 0,6,60,48\rangle & \langle 0,29,65,12\rangle \\ \langle 0,77,51,59\rangle & \langle 1,77,34,14\rangle & \langle 1,32,17,48\rangle \\ \langle 1,41,74,29\rangle & \langle 1,50,22,73\rangle & \langle 1,49,23,40\rangle \\ \langle 2,77,45,29\rangle & \langle 2,11,67,10\rangle & \langle 0,25,69,27\rangle \\ \langle 1,62,38,66\rangle & \langle 1,26,68,58\rangle & \langle 1,4,10,19\rangle \\ \langle 0,28,39,52\rangle. & & \end{array}$$

■

Proposition 7.13: $A_4(n, 6, [2, 1, 1]) = U(n, 6, [2, 1, 1])$ for each $n \in \{6, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 35, 37, 43, 55, 67, 79, 103\}$.

Proof: For each $n \in \{6, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 35, 37, 43, 55, 67, 79, 103\}$, $\mathcal{C}_n$ is an optimal $(n, 6, [2, 1, 1])_4$-code with size $U(n, 6, [2, 1, 1])$, where $\mathcal{C}_n$ is the set of cyclic (or quasi-cyclic) shifts of the vectors generated by the following vectors respectively.

1) $n = 6,\ m = 1,\ s = 1,\ M = 2$

$$P:\langle 0,1,2,3\rangle$$

2) $n = 8,\ m = 1,\ s = 1,\ M = 1$

$$P:\langle 0,1,3,5\rangle$$

3) $n = 10,\ m = 1,\ s = 1,\ M = 2$

$$P:\langle 0,1,2,8\rangle \quad \langle 0,4,7,9\rangle \quad \langle 1,3,6,7\rangle$$

4) $n = 11,\ m = 3,\ s = 3,\ M = 11$

$$\begin{array}{lll} P:\langle 1,2,3,4\rangle & \langle 1,5,6,7\rangle & \langle 2,6,5,10\rangle \\ R:\langle 2,8,9,7\rangle & \langle 0,10,4,5\rangle & \langle 4,8,1,0\rangle \\ \langle 3,5,0,8\rangle & \langle 8,10,6,3\rangle & \langle 0,9,6,2\rangle \\ \langle 4,5,2,9\rangle & & \end{array}$$

5) $n = 13,\ m = 7,\ s = 2,\ M = 1$

$$P:\langle 0,1,4,10\rangle$$

6) $n = 14,\ m = 1,\ s = 1,\ M = 1$

$$P:\langle 0,2,3,9\rangle \quad \langle 0,6,5,10\rangle$$

7) $n = 16,\ m = 1,\ s = 1,\ M = 2$

$$\begin{array}{lll} P:\langle 0,1,2,3\rangle & \langle 0,4,9,15\rangle & \langle 0,6,13,14\rangle \\ \langle 1,5,10,12\rangle & \langle 1,11,9,14\rangle & \end{array}$$

8) $n = 17$, $M = 17$, the code is given in Table I.

9) $n = 19,\ m = 5,\ s = 3,\ M = 1$

$$P:\langle 0,1,8,12\rangle$$

10) $n = 22,\ m = 3,\ s = 3,\ M = 2$

$$\begin{array}{lll} P:\langle 0,13,1,10\rangle & & \\ R:\langle 1,17,13,21\rangle & \langle 1,4,2,15\rangle & \langle 0,16,12,21\rangle \\ \langle 1,8,0,12\rangle & & \end{array}$$

11) $n = 23$, $M = 23$, the code is given in Table II.

12) $n = 25,\ m = 7,\ s = 2,\ M = 1$

$$P:\langle 0,1,3,23\rangle \quad \langle 0,8,20,13\rangle$$

13) $n = 28,\ m = 5,\ s = 5,\ M = 2$

$$\begin{array}{lll} P:\langle 0,1,3,6\rangle & & \\ R:\langle 1,5,12,17\rangle & \langle 0,24,14,17\rangle & \langle 0,16,8,23\rangle \\ \langle 1,9,2,15\rangle & & \end{array}$$

14) $n = 31,\ m = 2,\ s = 5,\ M = 1$

$$P:\langle 0,1,6,14\rangle$$

15) $n = 34,\ m = 7,\ s = 5,\ M = 2$

$$\begin{array}{lll} P:\langle 0,3,14,23\rangle & & \\ R:\langle 0,32,10,24\rangle & \langle 1,3,11,20\rangle & \langle 0,16,13,20\rangle \\ \langle 0,27,17,28\rangle & \langle 1,19,7,22\rangle & \langle 1,16,15,31\rangle \end{array}$$

16) $n = 35$, $M = 35$, the code is given in Table III.

17) $n = 37,\ m = 5,\ s = 6,\ M = 1$

$$P:\langle 0,1,11,27\rangle$$

18) $n = 43,\ m = 4,\ s = 7,\ M = 1$

$$P:\langle 0,1,7,13\rangle$$

19) $n = 55,\ m = 7,\ s = 2,\ M = 1$

$$\begin{array}{lll} P:\langle 0,9,53,6\rangle & \langle 0,54,11,15\rangle & \langle 0,4,40,25\rangle \\ R:\langle 0,31,49,45\rangle & \langle 0,20,39,23\rangle & \langle 0,17,30,43\rangle \end{array}$$

20) $n = 67$, $m = 3$, $s = 11$, $M = 1$

$$P : \langle 0, 1, 13, 55 \rangle$$

21) $n = 79$, $m = 3$, $s = 13$, $M = 1$

$$P : \langle 0, 1, 24, 56 \rangle$$

22) $n = 103$, $m = 3$, $s = 17$, $M = 1$

$$P : \langle 0, 1, 7, 97 \rangle.$$

■

Proposition 7.14: $A_4(7, 6, [2, 1, 1]) = 4$.

Proof: The 4 required codewords are:

$\langle 0, 1, 2, 3 \rangle \quad \langle 0, 4, 5, 6 \rangle \quad \langle 2, 3, 4, 5 \rangle$
$\langle 5, 6, 1, 2 \rangle$.

■